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On the boundaries of strict pseudospectra ☆

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The boundary of the ordinary ε-pseudospectrum of a square matrix is contained in the boundary of the strict ε-pseudospectrum. This content relation may be strict in some cases.

1. Introduction

The boundary of the ordinary pseudospectrum of a square matrix A of level ε, denoted by $\partial\Lambda_\varepsilon(A)$, is contained in the boundary of the strict pseudospectrum of the same level

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$\partial\Lambda'_\varepsilon(A)$ (see Remark 3.2, p. 280 in [1]). In this paper we will prove that in general these boundaries are not equal.

An equivalent problem is to determine whether the function of $z \mapsto \sigma_n(zI_n - A)$ can have local maxima. Thus, we will show that a complex number $z_0 \in \partial\Lambda'_\varepsilon(A) \setminus \partial\Lambda_\varepsilon(A)$ if and only if the function $z \mapsto \sigma_n(zI_n - A)$ reaches a local maximum at z_0. As a result, we will prove that the function $z \mapsto \sigma_n(zI_n - A)$ can have local maxima.

On the other hand, both the ordinary pseudospectrum of a matrix A of level ε and its boundary are semialgebraic sets [6]. We will prove that this property is also true for the strict pseudospectrum. This fact will allow us to prove that the set $\partial\Lambda'_\varepsilon(A) \setminus \partial\Lambda_\varepsilon(A)$ can be: empty, finite, or formed by the union of a finite set and a real analytic submanifold of dimension 1 with a finite number of connected components.

2. Previous notation and main results

For the inclusion relation between two sets X and Y we will use the notations $X \subset Y$ and $X \subsetneq Y$ to mean "X is contained in or equal to Y" and "X is strictly contained in Y", respectively. Let $\mathbb{C}^{n\times n}$ denote the space of $n \times n$ complex matrices. For any matrix $M \in \mathbb{C}^{n\times n}$ let

$$\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$$

denote its singular values in decreasing order. Let $\Lambda(A)$ denote the spectrum of the matrix $A \in \mathbb{C}^{n\times n}$. Given $A \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$ the *ordinary pseudospectrum of level* ε is the set

$$\Lambda_\varepsilon(A) := \bigcup_{\|\Delta\|\leq\varepsilon} \Lambda(A + \Delta),$$

where $\|\cdot\|$ denotes the spectral norm. Analogously, the *strict pseudospectrum of level* ε is the set

$$\Lambda'_\varepsilon(A) := \bigcup_{\|\Delta\|<\varepsilon} \Lambda(A + \Delta).$$

Let g denote the function

$$g(z) := \sigma_n(zI_n - A), \quad z \in \mathbb{C}; \tag{1}$$

using this function a characterization of the pseudospectra is given by

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} : g(z) \leq \varepsilon\}, \tag{2}$$

and

$$\Lambda'_\varepsilon(A) = \{z \in \mathbb{C} : g(z) < \varepsilon\}. \tag{3}$$