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Nonhyperbolic step skew-products: Ergodic approximation ☆

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Abstract

We study transitive step skew-product maps modeled over a complete shift of k, $k \geq 2$, symbols whose fiber maps are defined on the circle and have intermingled contracting and expanding regions. These dynamics are genuinely nonhyperbolic and exhibit simultaneously ergodic measures with positive, negative, and zero exponents.

We introduce a set of axioms for the fiber maps and study the dynamics of the resulting skew-product. These axioms turn out to capture the key mechanisms of the dynamics of nonhyperbolic robustly transitive maps with compact central leaves.

Focusing on the nonhyperbolic ergodic measures (with zero fiber exponent) of these systems, we prove that such measures are approximated in the weak$*$ topology and in entropy by hyperbolic ones. We also prove that they are in the intersection of the convex hulls of the measures with positive fiber exponent and with negative fiber exponent. Our methods also allow us to perturb hyperbolic measures. We can perturb a measure with negative exponent directly to a measure with positive exponent (and vice-versa), however we lose some amount of entropy in this process. The loss of entropy is determined by the difference between the Lyapunov exponents of the measures.

MSC: 37D25; 37D35; 37D30; 28D20; 28D99

Keywords: Entropy; Ergodic measures; Lyapunov exponents; Skew-product; Transitivity

ergodic measures
perturb easily
exponents rise or decrease

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1. Introduction

The aim of this paper is to understand the general structure and finer properties of the space of invariant measures of robustly transitive and robustly nonhyperbolic dynamical systems. For a large class[1] of such skew-products we approximate in entropy and in the weak∗ topology ergodic measures which are nonhyperbolic (with a zero Lyapunov exponent) and have positive entropy by measures supported on hyperbolic horseshoes, see Theorem 1. This result can be viewed as a nonhyperbolic version of a classical result by Katok[2] and also as a partial answer to a question about abundance of hyperbolicity posed by Buzzi in [11, Section 1.5].[3] As a consequence of our main results, in our setting, Theorem 1 can be read as follows: the intersection of the closed convex hull of ergodic measures with negative fiber exponent and the closed convex hull of ergodic measures with positive fiber exponent is non-empty and contains all ergodic measures with zero exponent, see Corollary 2.

Our results are a step of a program to understand the measure spaces, ergodic theory, and multifractal properties of general systems (diffeomorphisms, skew-product maps). As this at the present state of the art is far too ambitious in this vast generality, one may aim for gradually less specific classes of systems.[4] We focus on partially hyperbolic systems. The simplest, but still extremely complex, case occurs when the partially hyperbolic system has a central direction which is one-dimensional. Simplifying even one more step, in the case of partially hyperbolic diffeomorphisms, one may assume that the central bundle is integrable. In this case, three different scenario can occur: there exist only non-compact leaves (DA – derived from Anosov – diffeomorphisms), there exist simultaneously compact and non-compact leaves (time-1 maps of Anosov flows), or there exist only compact central leaves. The latter, and in some sense easiest, of these cases – compact central leaves – is still extremely rich (see, for instance, the pathological behaviors of the central foliations in [26,29]). On the other hand, using ingredients of one-dimensional dynamics, in this case one often has a very precise picture of the dynamics (see, for instance, [28,20]). As further simplification, we will restrict ourselves to step skew-products over a complete shift with circle-fiber maps. We hope that one will be able to gradually carry this program to more general settings. In fact, it turns out that the systems studied in this paper cover already typical robustly transitive and nonhyperbolic skew-products (see Section 8.3).

Besides the fact that skew-products as a class of systems have an intrinsic interest (there is a vast literature about different aspects, we do not go into further details here), they can also serve as a first step on the way to understand general types of dynamics of diffeomorphisms or endomorphisms. They also allow us to study essential aspects of a problem while escaping technical difficulties and this way enable us to study the problem in various steps of increasing difficulty.

To be a bit more precise, still in the partially hyperbolic setting with a nonhyperbolic central direction, when aiming for general systems, one is confronted with several problems of completely different nature and origin. First, restricting to systems with a one-dimensional central fiber enables us to study relatively easily their Lyapunov exponents which turn into Birkhoff averages of *continuous* functions, while in the general case they are provided by the Oseledets theorem and are measurable functions only. Moreover, in this case there is no entropy generated by the fiber dynamics (for details see Appendix). A second problem is the nonhyperbolicity reflected by the coexistence of hyperbolic measures and, consequently, of hyperbolic periodic points with different behavior in the central direction. Finally, there are problems related to the existence and regularity of the central foliations. Restricting our consideration to skew-products allows us to focus on the difficulty arising from the nonhyperbolicity, while escaping from the latter one. This approach also allows us to present our constructions (e.g. the multi-variable-time horseshoes and their symbolic extensions) in a transparent way. This strategy also allows us to establish an axiomatic approach, which is in fact completely justified and turns out to reflect quite well the general features of robustly transitive and nonhyperbolic systems.

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1 Open and dense for C^1 step skew-products and dense for general C^1 skew-products.

2 The result of Katok claims that any ergodic hyperbolic measure can be weak* and in entropy approximated by horseshoes. See [17,18] for $C^{1+\alpha}$ diffeomorphisms and also extensions in the context of C^1 diffeomorphisms with a dominated splitting in [12,23,15].

3 A bit more precisely, his question is the following: Among partially hyperbolic diffeomorphisms with one-dimensional center direction, are those with "enough" hyperbolic measures C^1 or C^2 dense?

4 An example of this strategy can be found by the line of papers studying, in the same context, the construction of nonhyperbolic ergodic measures: [16] (step skew-products), [19] (skew-products and some specific open sets of diffeomorphisms), [14,6] (generic diffeomorphisms), and [3] (settling open and densely the case of general diffeomorphisms).

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