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Codimension two surfaces pinched by normal curvature evolving by mean curvature flow

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Abstract

We prove that codimension two surfaces satisfying a nonlinear curvature condition depending on normal curvature smoothly evolve by mean curvature flow to round points.

Keywords: Curvature flow; Mean curvature; Nonlinear parabolic equations

1. Introduction

We consider two dimensional surfaces of codimension two immersed in Euclidean four-space, which includes, for example, the Clifford torus when viewed as a submanifold of $\mathbb{R}^4$. The main theorem we present asserts that surfaces satisfying a curvature pinching depending on normal curvature are deformed by the mean curvature flow to round points. In contrast to hypersurfaces, very little progress has been made on mean curvature flow in high codimension owing to the nontrivial structure of the normal bundle. The best result to date is due to Andrews and Baker [1], where it is shown that, for suitable values of a constant k depending on dimension but not codimension, submanifolds satisfying the pinching condition $|A|^2 \leq k|H|^2$ evolve under the mean curvature flow to round points, which can be considered a high codimension analogue of Huisken's seminal result on mean curvature flow of hypersurfaces [5]. In this paper we show for the first time that inclusion of normal curvature in the pinching cone enables improved geometric estimates, expanding the class of surfaces known to be diffeomorphic to round spheres.

The submanifold estimates are much more difficult than their hypersurface counterparts, being complicated by the presence of normal curvature. The main theorem of [1] is optimal for submanifolds of dimension four and greater (independent of the codimension), where the tori $\mathbb{S}^{n-1}(\epsilon) \times \mathbb{S}(1) \subset \mathbb{R}^n \times \mathbb{R}^2$ are obstructions to improving the pinching

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constant beyond $1/(n-1)$. The theorem is suboptimal in dimensions two and three, with pinching constant $k = 4/(3n)$, because of unfavourable reaction terms. With the inclusion of normal curvature, the new pinching condition turns out to be optimal for the reaction terms, but the gradient terms still obstruct the attainment of optimal pinching, similar to the flow of hypersurfaces in a spherical background [6]. The main result we obtain in this article is as follows:

Theorem 1.1. *Suppose $\Sigma_0 = F_0(\Sigma^2)$ is a closed surface smoothly immersed in $\mathbb{R}^4$. If Σ_0 satisfies $|H|_{min} > 0$ and $|A|^2 + 2\gamma|K^\perp| \le k|H|^2$, where $\gamma = 1 - 4/3k$ and $k \le 29/40$, then the mean curvature flow of Σ_0 has a unique smooth solution Σ_t on a finite maximal time interval $t \in [0, T)$. There exists a sequence of rescaled mean curvature flows $F_j : \Sigma^2 \times I_j \to \mathbb{R}^4$ containing a subsequence of mean curvature flows (also indexed by j) that converges to a limit mean curvature flow $F_\infty : \Sigma^2_\infty \times (-\infty, 0] \to \mathbb{R}^4$ on compact sets of $\mathbb{R}^4 \times \mathbb{R}$ as $j \to \infty$. Moreover, the limit mean curvature flow is a shrinking sphere.*

This theorem improves the pinching constant of [1] from $2/3$ to $3/4 - 1/40$, which is, similar to the hypersurface theory, almost the best constant thought to be achievable with the mean curvature flow. The inclusion of normal curvature in the pinching condition cancels the unfavourable reaction terms encountered in [1], however, the gradient of the normal curvature prohibits pushing the pinching constant all the way to $3/4$. We conjecture that the Clifford torus viewed as a two-surface of codimension two in $\mathbb{R}^4$ is the true obstruction to the theorem, corresponding to an optimal pinching constant of $k = 1$. The Clifford torus is still intrinsically flat but no longer minimal in $\mathbb{R}^4$ and satisfies $|A|^2 = |H|^2$.

We take this opportunity to announce another new result of independent interest, discovered in the course of estimating the nonlinearity in the Simons identity (see Proposition 5.3). Obtaining a positive lower bound on the nonlinearity in Simons' identity is a crucial step in the integral estimates used to prove convergence to a round point. In the case of two-surfaces of codimension two (in this case immersed in a Euclidean background), it is possible to compute the nonlinearity exactly with the result that $Z = 2K|\mathring{A}|^2 - 2|K^\perp|^2$. The Simons identity plays a key role in a series of classification results initiated in a famous paper by Chern, do Carmo and Kobayashi [3], where they prove that if a n-dimensional submanifold of a $(n+p)$-dimensional sphere satisfies $|A|^2 \le n/(1 - 1/p)$, then the submanifold is totally geodesic, or if the equality holds identically, then it is the Clifford torus or Veronese surface. With our refined understanding of the Simons identity nonlinearity we are able to provide a new classification result depending not on the length of the second fundamental form, but rather on a pointwise pinching of the intrinsic and normal curvatures.

Theorem 1.2. *Suppose a two surface Σ^2 minimally immersed in $\mathbb{S}^4$ satisfies $|K^\perp| \le 2|K|$. Then either*

i) $|A|^2 \equiv 0$ *and the surface is a geodesic sphere; or*
ii) $|A|^2 \not\equiv 0$, *in which case either*
   (a) $|K^\perp| = 0$ *and the surface is the Clifford torus, or*
   (b) $K^\perp \ne 0$ *and it is the Veronese surface.*

We intend to investigate the motion of submanifolds of a sphere in a sequel to this paper, where a proof of the above theorem more naturally resides. The argument involves careful examination of the curvature terms and an application of the strong maximum principle.

2. Notation and preliminary results

We adhere to the notation of [1] and in particular use the canonical space–time connections introduced in that paper. A fundamental ingredient in the derivation of the evolution equations is Simons' identity:

$$\Delta h_{ij} = \nabla_i \nabla_j H + H \cdot h_{ip} h_{pj} - h_{ij} \cdot h_{pq} h_{pq} + 2h_{jq} \cdot h_{ip} h_{pq} - h_{iq} \cdot h_{qp} h_{pj} - h_{jq} \cdot h_{qp} h_{pi}. \tag{1}$$

The timelike Codazzi equation combined with Simons' identity produces the evolution equation for the second fundamental form:

$$\nabla_{\partial_t} h_{ij} = \Delta h_{ij} + h_{ij} \cdot h_{pq} h_{pq} + h_{iq} \cdot h_{qp} h_{pj} + h_{jq} \cdot h_{qp} h_{pi} - 2h_{ip} \cdot h_{jq} h_{pq}. \tag{2}$$

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