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Ground states of pseudo-relativistic boson stars under the critical stellar mass

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Abstract

We consider ground states of pseudo-relativistic boson stars with a self-interacting potential $K(x)$ in $\mathbb{R}^3$, which can be described by minimizers of the pseudo-relativistic Hartree energy functional. Under some assumptions on $K(x)$, minimizers exist if the stellar mass N satisfies $0 < N < N^*$, and there is no minimizer if $N > N^*$, where N^* is called the critical stellar mass. In contrast to the case of the Coulomb-type potential where $K(x) \equiv 1$, we prove that the existence of minimizers may occur at $N = N^*$, depending on the local profile of $K(x)$ near the origin. When there is no minimizer at $N = N^*$, we also present a detailed analysis of the behavior of minimizers as N approaches N^* from below, for which the stellar mass concentrates at a unique point.

Keywords: Ground states; Boson stars; Mass concentration; Critical mass

1. Introduction

We study ground states of pseudo-relativistic boson stars in the mean field limit, which can be described (cf. [5,9,20]) by minimizers of the following variational problem

$$e(N) := \inf\left\{\mathcal{E}(u) : u \in H^{\frac{1}{2}}(\mathbb{R}^3) \text{ and } \int_{\mathbb{R}^3} |u(x)|^2 dx = N\right\} \tag{1.1}$$

under the stellar mass $N > 0$ of boson stars, where the pseudo-relativistic Hartree energy functional $\mathcal{E}(u)$ is defined by

$$\mathcal{E}(u) := \int_{\mathbb{R}^3} \bar{u}\left(\sqrt{-\Delta + m^2} - m\right)u dx - \frac{1}{2}\int_{\mathbb{R}^3} \left(\frac{K(x)}{|x|} * |u|^2\right)|u|^2 dx. \tag{1.2}$$

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Here the operator $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{k^2 + m^2}$ in Fourier space, which describes the kinetic and rest energy of a relativistic particle with rest mass $m > 0$, and the symbol $*$ stands for the convolution on $\mathbb{R}^3$. Moreover, we always assume that the self-interacting potential $K(x)$ of boson stars satisfies $0 < K(x) \le K(0) = 1$ in $\mathbb{R}^3$. From physical point of view, there are two typical types of such potentials: one is the Coulomb-type potential $K(x) \equiv 1$, and the other is the Yukawa-type potential $K(x) \equiv e^{-\mu|x|}$ with $\mu > 0$. Our investigation of $e(N)$ is motivated by our recent series of works [4,12–15], where we studied the minimization problem of the Laplacian type arising from attractive Bose–Einstein condensates (BEC). However, the analysis of $e(N)$ is more complicated in a substantial way, due to the nonlocal nature of the pseudo-differential operator $\sqrt{-\Delta + m^2}$, and the convolution-type nonlinearity as well.

In the recent works [8–10,16,17,20] and references therein, the authors discussed $e(N)$ for the case of the Coulomb-type potential, i.e., $K(x) \equiv 1$, where the existence, nonexistence, dynamics and some other analytic properties of minimizers for $e(N)$ were obtained. The existing results of analyzing $e(N)$ show that the problem (1.1) is related well to the following Gagliardo–Nirenberg type inequality

$$\int_{\mathbb{R}^3} \Big(\frac{1}{|x|} * |u|^2\Big)|u|^2 dx \le \frac{2}{\|Q\|_2^2} \|(-\Delta)^{1/4} u\|_2^2 \, \|u\|_2^2, \quad u \in H^{\frac{1}{2}}(\mathbb{R}^3), \tag{1.3}$$

where $Q(x) = Q(|x|) > 0$ is a ground state, up to translations and suitable rescaling (cf. [2,6,9]), of the fractional equation

$$\sqrt{-\Delta}\, u + u - \Big(\frac{1}{|x|} * |u|^2\Big) u = 0 \ \text{ in } \ \mathbb{R}^3, \quad \text{where } \ u \in H^{\frac{1}{2}}(\mathbb{R}^3). \tag{1.4}$$

Similar to that of [3,6,7,11], here we define

Definition 1.1. If $Q(x) = Q(|x|) \in H^{\frac{1}{2}}(\mathbb{R}^3)$ is a positive solution of (1.4), and $Q(x)$ optimizes (1.3), we then say that $Q(x)$ is a ground state of (1.4).

Note from [1,6,9] that (1.4) admits ground states, and all ground states of (1.4) must be radially symmetric and nonincreasing. Therefore, one can define the nonempty set $\mathcal{G}$ by

$$\mathcal{G} = \big\{ Q(x) = Q(|x|) > 0 : Q(x) \text{ is a ground state of (1.4)} \big\}. \tag{1.5}$$

It thus follows from (1.3) and (1.4) that

$$\|Q\|_2^2 = \|(-\Delta)^{1/4} Q\|_2^2 = \frac{1}{2} \int_{\mathbb{R}^3} \Big(\frac{1}{|x|} * Q^2\Big) Q^2 dx \ \text{ for all } Q \in \mathcal{G}. \tag{1.6}$$

Moreover, one can derive from [6, Lemma 2.2] that there exists $C > 0$ such that for all $Q(x) \in \mathcal{G}$,

$$|Q(x)| \le C(1 + |x|)^{-4} \ \text{ in } \mathbb{R}^3, \tag{1.7}$$

and

$$(|x|^{-1} * |Q|^2)(x) \le C(1 + |x|)^{-1} \ \text{ in } \mathbb{R}^3. \tag{1.8}$$

Making full use of the above results, inspired by [9,12,20] we first derive the following existence of a critical stellar mass N^*.

Theorem 1.1. *Let Q be a ground state of (1.4), and suppose that $m > 0$ and*

$$K(x) \in C(\mathbb{R}^3) \text{ satisfies } 0 < K(x) \le K(0) = 1 \text{ in } \mathbb{R}^3. \tag{1.9}$$

Then,

(a). If $0 < N < N^* := \|Q\|_2^2$, and assume that

$$\text{either } m > 0 \text{ is large sufficiently or } K(x) \ge O(|x|^{-\alpha}) \text{ as } |x| \to \infty \text{ with } 0 < \alpha < 1, \tag{1.10}$$

then there exists at least one minimizer for (1.1).

(b). If $N > N^*$, there is no minimizer for (1.1).

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