ARTICLE IN PRESS
ANIHPC:2793

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ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Convex billiards on convex spheres

Pengfei Zhang

Department of Mathematics, University of Mississippi, Oxford, MS 38677, United States

Received 26 May 2015; received in revised form 17 February 2016; accepted 7 July 2016

Abstract

In this paper we study the dynamical billiards on a convex 2D sphere. We investigate some generic properties of the convex billiards on a general convex sphere. We prove that C^∞ generically, every periodic point is either hyperbolic or elliptic with irrational rotation number. Moreover, every hyperbolic periodic point admits some transverse homoclinic intersections. A new ingredient in our approach is Herman's result on Diophantine invariant curves that we use to prove the nonlinear stability of elliptic periodic points for a dense subset of convex billiards.
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MSC: 37D40; 37D50; 37C20; 37E40

Keywords: Convex billiards; Generic properties; Homoclinic intersections; Diophantine number; Nonlinearly stable

1. Introduction

The dynamical billiards, as a class of dynamical systems, were introduced by Birkhoff [5,6] in his study of Lagrangian systems with two degrees of freedom. A Lagrangian system with two degrees of freedom is isomorphic with the motion of a mass particle moving on a surface rotating uniformly about a fixed axis and carrying a fixed conservative field of force with it. If the surface is not rotating and the force vanishes, then the particle moves along geodesics on the surface. If the surface has boundary, then the resulting system is a billiard system.

The classical results of dynamical billiards are closely related to geometrical optics, which has a much longer history. For example, the discovery of the integrability of elliptic billiards, according to Sarnak [49], goes back at least to Boscovich in 1757. Surprisingly, the billiard dynamics is also related to the spectra property of Laplace–Beltrami operator on manifolds with a boundary. More precisely, Weyl's law in spectral theory gives the first order asymptotic distribution of eigenvalues of the Laplace–Beltrami operator on a bounded domain. Weyl's conjecture on the second order asymptotic distribution was proved by Ivrii [29] for any compact manifold with boundary, under the assumption that the measure of periodic points of billiard dynamics on that manifold is zero.

E-mail address: pzhang2@olemiss.edu.

http://dx.doi.org/10.1016/j.anihpc.2016.07.001
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Please cite this article in press as: P. Zhang, Convex billiards on convex spheres, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2016.07.001

Current study of dynamical billiard systems mainly focuses on the Euclidean case. Birkhoff studied the dynamical billiards inside a convex domain on the plane. Birkhoff also conjectured that ellipses are the only integrable billiards. A weak version of this conjecture was proved by Bialy [3]. The dynamical billiards on a bounded domain with convex scatterers were introduced by Sinai in his study of Boltzmann Ergodic Hypothesis [50] on ideal gases. Sinai discovered the dispersing mechanism and proved that dispersing billiards are hyperbolic and ergodic. Since then, the mathematical and physical study of chaotic billiards has developed at a remarkable speed (see [14]), particularly after the various defocusing mechanisms discovered by Bunimovich [9,10], Wojtkowski [55], Markarian [33] and Donnay [21]. Very recently, the dynamics of some asymmetric lemon billiards are proved to be hyperbolic [12], for which the separation condition in the defocusing mechanism was strongly violated. See [53,30,28] for the study of chaotic billiards on general surfaces. The study of chaotic billiards also provides the key idea for the construction of hyperbolic geodesic flows on S^2, see [19,20,13].

Dynamical billiards on curved surfaces are related to the study of quantum magnetic confinement of non-planar 2D electron gases (2DEG) in semiconductors [25], where the effect of varying the curvature of the surface corresponds to a change in the potential energy of the system. The dynamical billiards can be viewed as a mathematical model for this system, and may be used to investigate the electron transport properties of the semiconductors. As mentioned in [28], the advances in semiconductor fabrication techniques allow to manufacture solid state (mesoscopic) devices where electrons are confined to curved surfaces.

In this paper we consider the convex billiards on convex spheres. Recall that the 2D sphere S^2 with a smooth Riemannian metric g is said to be (strictly) *convex*, if it has positive Gaussian curvature: $K_g(x) > 0$ for all $x \in S^2$. Given a tangent vector $\mathbf{v} \in T_x S^2$, the geodesic passing through x in the direction of $\mathbf{v}$ is defined by the exponential map $\gamma_{\mathbf{v}} : \mathbb{R} \to S^2$, $t \mapsto \exp_x(t\mathbf{v})$. For any two points $p, q \in S^2$, let $d(p, q)$ be the length of the shortest geodesics connecting p and q. Let $\mathrm{Inj}(S^2, g)$ be the injective radius of (S^2, g).

Example. Let S^2 be the unit sphere in $\mathbb{R}^3$ endowed with the round metric g_0. Then $K_0 \equiv 1$, and every geodesic on S^2 moves along a great circle. Let $p, q \in S^2$ be two points on the sphere, and α be the angle between the two position vectors $\mathbf{p}, \mathbf{q}$. Then the geodesic distance $d_0(p, q)$ between p and q is given by $d_0(p, q) = \alpha(\mathbf{p}, \mathbf{q})$, and $\cos\alpha = \langle \mathbf{p}, \mathbf{q} \rangle$. Therefore, $d_0(p, q) = \arccos\langle \mathbf{p}, \mathbf{q} \rangle$. Moreover, $\mathrm{Inj}(S^2, g_0) = \pi$. The dynamical billiards inside convex subsets of (S^2, g_0) have been studied recently in [8,4,16]. Regarding the Ivrii conjecture, it is proved in [7] that the set of periodic points of period 3 has zero measure for *any* billiard on the unit sphere.

Definition 1.1. Let (S^2, g) be a convex sphere. A closed subset $Q \subset S^2$ is said to be (geodesically) *convex*, if Q is simply connected, and for any two points $x, y \in Q$, there is a unique minimizing geodesic contained in Q connecting x and y. A convex domain Q is said to be *strictly convex*, if the interior of each minimizing geodesic is contained in the interior Q^o of Q.

Let $Q \subset S^2$ be a convex domain, s be the arc-length parameter of $\Gamma = \partial Q$, and $\kappa(s)$ be the geodesic curvature of Γ at $\Gamma(s)$. Note that $\kappa(s) \geq 0$ for all s. If Q is strictly convex, then $\kappa(s) > 0$ for all s (except on a closed set without interior). By definition, there are no conjugate points inside a convex domain Q. In the following we require that there are no conjugate points on the closed domain Q. A sufficient condition for nonexistence of conjugate point is that $\mathrm{diam}(Q) < \mathrm{Inj}(S^2, g)$.

The dynamical billiard on Q can be defined analogously to the planar case. That is, a particle moves along geodesics inside Q, and reflects elastically upon hitting the boundary ∂Q. Suppose the previous reflection happens at $\Gamma(s)$. Let θ be the angle measured from the (positive) tangent direction $\dot{\Gamma}(s)$ to the post-reflection velocity of that particle. Then the *billiard map* F sends (s, θ) to the next reflection (s_1, θ_1) with ∂Q. The *phase space* of the billiard map F on Q is given by $M = \Gamma \times (0, \pi)$. Note that the 2-form $\omega = \sin\theta\, ds \wedge d\theta$ is a symplectic form on M. Let μ be the smooth probability measure on M with density $d\mu = \frac{1}{2|\partial Q|} \sin\theta\, ds\, d\theta$.

Theorem 1. *Let (S^2, g) be a convex sphere and $Q \subset S^2$ be a strictly convex domain with C^r smooth boundary $\Gamma = \partial Q$. Then billiard map $F : M \to M$ is a symplectic twist map. In particular, F preserves the measure μ.*

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