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Existence of solutions to a two-dimensional model for nonisothermal two-phase flows of incompressible fluids

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Abstract

We consider a thermodynamically consistent diffuse interface model describing two-phase flows of incompressible fluids in a non-isothermal setting. The model was recently introduced in [11] where existence of weak solutions was proved in three space dimensions. Here, we aim to study the properties of solutions in the two-dimensional case. In particular, we can show existence of global in time solutions satisfying a stronger formulation of the model with respect to the one considered in [11].

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Keywords: Cahn–Hilliard; Navier–Stokes; Incompressible non-isothermal binary fluid; Global-in-time existence; A-priori estimates

1. Introduction

We consider here a mathematical model for two-phase flows of non-isothermal incompressible fluids in a bounded container Ω in $\mathbb{R}^2$ during a finite time interval $(0, T)$, with no restrictions on the magnitude of the final time $T > 0$. The model consists of a PDE system describing the evolution of the unknown variables $\boldsymbol{u}$ (macroscopic velocity), φ (order parameter), μ (chemical potential), ϑ (absolute temperature), and it takes the form

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$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.1}$$

$$\boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u} - \operatorname{div}(\nabla\varphi \otimes \nabla\varphi), \tag{1.2}$$

$$\varphi_t + \boldsymbol{u} \cdot \nabla\varphi = \Delta\mu, \tag{1.3}$$

$$\mu = -\Delta\varphi + F'(\varphi) - \vartheta, \tag{1.4}$$

$$\vartheta_t + \boldsymbol{u} \cdot \nabla\vartheta + \vartheta\big(\varphi_t + \boldsymbol{u} \cdot \nabla\varphi\big) - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = |\nabla \boldsymbol{u}|^2 + |\nabla\mu|^2. \tag{1.5}$$

Relation (1.2), with the incompressibility constraint (1.1), represents a variant of the Navier–Stokes system; (1.3)–(1.4) correspond to a form of the Cahn–Hilliard system [10] for phase separation; (1.5) is the internal energy equation describing the evolution of temperature. Note that transport effects are admitted for all variables in view of the occurrence of material derivatives in (1.2), (1.3), and (1.5). As usual, the variable p in the Navier–Stokes system (1.2) represents the (unknown) pressure. The function F whose derivative appears in (1.4) is a possibly nonconvex potential whose minima represent the least energy configurations of the phase variable. Here we will assume that F is smooth and has at most a power-like growth at infinity. Indeed, it is not clear whether our result could be extended to other classes of physically significant potential, having nonsmooth or singular character (like, e.g., the so-called logarithmic potential $F(r) = (1+r)\log(1+r) + (1-r)\log(1-r) - r^2$ which typically appears in Cahn–Hilliard-based models, see, e.g., [30]). Finally, the coefficient $\kappa(\vartheta)$ in (1.5) stands for the heat conductivity of the fluid. Here we shall assume that κ grows at infinity like a sufficiently high power of ϑ (see (2.11) below).

The PDE system (1.1)–(1.4), in the case of a constant temperature ϑ, is referred to in the literature as "Model H". Even if many authors considered the isothermal Model H (cf., for instance, [3,19–23,25] for the physical derivation of the model, and [1,2,4,5,15–17,27,34,41,42] for the study of the resulting evolution system), up to our knowledge no contributions are so far present in the literature in the non-isothermal case, except for [35], where a linearization of (1.5) is considered, [18], where the numerical analysis of a phase-field model for binary quasi-incompressible fluid with thermocapillarity effects is performed, [40], where a Cahn–Hilliard–Boussinesq system has been studied, and our previous paper [11], where system (1.1)–(1.5) has been analyzed in the 3D case.

Actually, the above model was proposed in our recent work [11] starting from the balance laws for internal energy and entropy; in particular, thermodynamical consistence was shown to hold for any (positive) value of the absolute temperature ϑ. Moreover, existence of solutions for a *weak formulation* of (1.1)–(1.5) was proved when the system is settled in a smooth bounded domain $\Omega \subset \mathbb{R}^3$ and complemented with no-flux conditions for φ, μ and ϑ and with *slip* conditions for $\boldsymbol{u}$. Mathematically, the main source of difficulty in system (1.1)–(1.5) comes from the quadratic terms on the right hand side of (1.5). Their occurrence is physically motivated as one considers the derivation of the model in terms of the energy and entropy balances (cf. [11, Sec. 2]). Roughly speaking, one can say that these terms represent a source of thermal energy coming from the dissipation of kinetic energy due to viscosity (cf. (1.2)) and of configuration energy due to action of micro-forces (cf. (1.3)–(1.4)). This energy dissipation, as expected, happens in such a way to increase the entropy of the system.

From the analytical viewpoint, the quadratic terms in (1.5) can be controlled only in the L^1-norm, at least in the 3D-case. This is the reason why in [11] we introduced a suitable weak formulation along the lines of an idea originally developed in [8,12] for dealing with heat conduction in fluids, in [13] for solid-liquid phase transitions, and more recently in [32] for damage phenomena. In such a setting, the "heat" equation (1.5) is replaced with a relation describing the balance of *total energy* (i.e., not only of thermal energy), which does no longer contain quadratic terms. This is complemented with a distributional version of the *entropy inequality*.

Looking at the 2D model, whose analysis is the aim of this paper, it is well-known that, for the Navier–Stokes system (1.2), additional regularity is available provided that the forcing term (here given by $-\operatorname{div}(\nabla\varphi \otimes \nabla\varphi)$) lies in L^2 (cf., e.g., [31]). Fortunately, this seems to happen in our case, as one can readily check starting from the available energy and entropy estimates; hence, there is hope to get additional summability for the quadratic term $|\nabla \boldsymbol{u}|^2$ in (1.5). This was the motivation which led us to investigate whether it is possible to prove existence of a solution to the *original* (strong) system (1.1)–(1.5) in two space dimensions.

In order to explain our ideas, we first need to introduce some basic assumptions. First of all, to avoid technical complications related with the choice of boundary data, we ask the system to be settled in the unit torus $\Omega := [0, 1] \times [0, 1]$ and complemented with periodic boundary conditions for all unknowns. It is worth noting that some other types of boundary conditions could be assumed. For instance, we may take, as in [11], no-flux (i.e., homogeneous Neumann) conditions for φ, μ and ϑ (as it is physically reasonable if one assumes the container Ω to be insulated

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