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Optimal regularity in the optimal switching problem

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Abstract

In this article we study the optimal regularity for solutions to the following weakly coupled system with interconnected obstacles

$$\begin{cases} \min(-\Delta u^1 + f^1, u^1 - u^2 + \psi^1) = 0 \\ \min(-\Delta u^2 + f^2, u^2 - u^1 + \psi^2) = 0, \end{cases}$$

arising in the optimal switching problem with two modes.

We derive the optimal $C^{1,1}$-regularity for the minimal solution under the assumption that the zero loop set $\mathscr{L} := \{\psi^1 + \psi^2 = 0\}$ is the closure of its interior. This result is optimal and we provide a counterexample showing that the $C^{1,1}$-regularity does not hold without the assumption $\mathscr{L} = \overline{\mathscr{L}^0}$.

Keywords: Optimal switching problem; Regularity theory; The obstacle problem; Double obstacle problem; Free boundary problem; Nonlinear elliptic system

1. Introduction

We consider the following system of weakly coupled equations of obstacle type

$$\begin{cases} \min(-\Delta u^1 + f^1, u^1 - u^2 + \psi^1) = 0 \\ \min(-\Delta u^2 + f^2, u^2 - u^1 + \psi^2) = 0, \end{cases} \tag{1}$$

with given Dirichlet boundary conditions $u^i = g^i$ on $\partial\Omega$. These type of systems arise in optimal switching problems with two switching modes. Here f^1 and f^2 are the running cost functions corresponding to the switching modes. The functions ψ^1 and ψ^2 are the costs of switching from one mode to the other. More details on the optimal switching problem are provided in Section 2.1.

The uniqueness and $C^{1,1}$-regularity of the solutions to such systems have been studied in the literature under the assumption that the switching costs are nonnegative constants, [4,7,2]. Obstacle type weakly coupled systems with first order Hamiltonians and nonconstant switching costs have been studied in [3,6]. In the paper [3], Section 5 the authors investigate the speed of convergence of the solutions to a penalized system, they also show that the solution

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of the first order Hamilton–Jacobi obstacle type system is Lipschitz continuous, under the assumption that each of the switching costs is bounded from below by a positive constant.

In our paper we make only the nonnegative loop assumption. This is a necessary condition for the system to be well-defined. Indeed, let (u^1, u^2) be a solution to (1), then $u^1 - u^2 + \psi^1 \geq 0$ and $u^2 - u^1 + \psi^2 \geq 0$, which implies

$$\psi^1(x) + \psi^2(x) \geq 0. \tag{2}$$

In the optimal switching setting, the condition (2) prevents the agent from making arbitrary gains by looping, in the sense that $\psi^1(x) + \psi^2(x)$ is the cost of switching from one mode to the other and immediately switching back. We denote the set where it is possible to switch for free by

$$\mathscr{L} = \{x \in \Omega \mid \psi^1(x) + \psi^2(x) = 0\},$$

and call it free switching or zero loop set.

By using the penalization/regularization method we derive the existence of solutions, showing that through a subsequence the solutions of the penalized system converge to the minimal solution (u_0^1, u_0^2) to (1). Then we see that the solution $u_0^i \in C^{1,\gamma}$, for every $0 < \gamma < 1$ and

$$\|\Delta u_0^i\|_{L^\infty(\Omega)} \leq \max_i \|\Delta \psi^i\|_{L^\infty(\Omega)} + 3 \max_i \|f^i\|_{L^\infty(\Omega)}. \tag{3}$$

The aim of the paper is to investigate if the solutions are $C^{1,1}$, which is the best regularity that we can hope that the solutions achieve. The structure of our system shows that at some subdomains of Ω, the regularity of the solutions can be derived by already known $C^{1,1}$-regularity results for the obstacle problem. In our discussion we see that the main point is to describe the regularity at so called meeting points lying on $\partial\mathscr{L}$, the boundary of the zero loop set.

In the main theorem, Theorem 4, we show that at the meeting points $x_0 \in \partial\mathscr{L}^0 \cap \Omega$ the solutions are $C^{2,\alpha}$, under the assumption that $f^i \in C^\alpha$ and $\psi^i \in C^{2,\alpha}$. By $\mathscr{L}^0$ we denote the interior of the set $\mathscr{L}$, and by pointwise $C^{2,\alpha}$ regularity we mean uniform approximation with a second order polynomial with the speed $r^{2+\alpha}$.

The idea of the proof is the same as in deriving the optimal regularity for the no-sign obstacle problem in [1]. The proof is based on the *BMO*-estimates for $D^2u_0^1$ and $D^2u_0^2$ following from the estimate (3). At the point x_0, we consider $r^{2+\alpha}$-th order rescalings of u_0^i denoted by v_r^i, and show that these are uniformly bounded in $W^{2,2}(B_1)$. Then, looking at the corresponding system for (v_r^1, v_r^2), we conclude that the rescalings are uniformly bounded in the ball B_1.

In the end we justify our assumption $0 \in \partial\mathscr{L}^0$ with a counterexample: We consider a particular system in $\mathbb{R}^2$, where the zero loop set $\mathscr{L} = \{0\}$, then we find an explicit solution, that is not $C^{1,1}$.

The paper is structured as follows: In Section 2 we provide some background material. In Section 3 we use the penalization method to derive the existence of strong solutions, and observe that these are actually minimal solutions. The main results are presented in the last section, where we prove that the minimal solution is locally $C^{1,1}$ if the zero loop set is the closure of its interior, and provide a counterexample to $C^{1,1}$-regularity when $\psi^1 + \psi^2$ has an isolated zero.

2. Background material

In this section we state some known results, which we use in our discussion, without giving any proofs.

2.1. Optimal switching problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. We consider an agent that can be anywhere in Ω and in one of a finite number m of states. For every $1 \leq i \leq m$, the agent moves in Ω according to a diffusion

$$dx = b_i(x)dt + \sigma_i(x)dW_t,$$

where W_t is a Brownian motion in a suitable probability space, $b_i : \Omega \to \mathbb{R}^n$ and $\sigma_i : \Omega \to \mathbb{R}^{n\times m}$ are smooth functions. The generator of the diffusions is denoted by $L^i v = \frac{1}{2}\sigma_i\sigma_i^T : D^2 v + b_i \cdot Dv$.

The agent can switch from any diffusion mode to another. At every instant t the agent pays a running cost $f^{i(t)}(x)$, depending on the present state $i(t)$ and position x. Additionally, when changing state i to state j he incurs in a switching cost $-\psi^{ij}(x)$. Finally, when the diffusion reaches the boundary and the agent is in state i, the process is stopped