ARTICLE IN PRESS
ANIHPC:2754

Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN ●●● (●●●●) ●●●–●●●

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Shape optimization problems with Robin conditions on the free boundary

Dorin Bucur [a,1], Alessandro Giacomini [b,*,2]

[a] *Laboratoire de Mathématiques, CNRS UMR 5127 Université de Savoie & Institut Universitaire de France, Campus Scientifique, 73376 Le-Bourget-Du-Lac, France*

[b] *DICATAM, Sezione di Matematica, Università degli Studi di Brescia, Via Valotti 9, 25133 Brescia, Italy*

Received 13 January 2015; accepted 19 July 2015

Abstract

We provide a free discontinuity approach to a class of shape optimization problems involving Robin conditions on the free boundary. More precisely, we identify a large family of domains on which such problems are well posed in a way that the extended problem can be considered a relaxed version of the corresponding one on regular domains, we prove existence of a solution and obtain some qualitative information on the optimal sets.

Keywords: Free boundary problems; Shape optimization problems; Robin boundary conditions; Free discontinuity problems; Functions of bounded variation

1. Introduction

Let $D \subseteq \mathbb{R}^d$ be a *design region* which we assume to be open, bounded and with a Lipschitz boundary. Consider $B \subset D$ open and with a C^1-boundary and $g \in C^1(\mathbb{R}^d)$ such that

$$0 < c_1 \leq g \leq c_2 \qquad \text{on } B.$$

The main concern of the paper is the following *shape optimization problem*.

* Corresponding author.
E-mail addresses: dorin.bucur@univ-savoie.fr (D. Bucur), alessandro.giacomini@unibs.it (A. Giacomini).

1 The work of D.B. was supported by the ANR-12-BS01-0014-01 Geometrya.

2 The work of A.G. was supported by the Italian Ministry of Education, University and Research under the project "Calculus of Variations" (PRIN 2010-11). He is also member of the Gruppo Nazionale per L'Analisi Matematica, la Probabilità e loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

http://dx.doi.org/10.1016/j.anihpc.2015.07.001
0294-1449/

Please cite this article in press as: D. Bucur, A. Giacomini, Shape optimization problems with Robin conditions on the free boundary, Ann. I. H. Poincaré – AN (2015), http://dx.doi.org/10.1016/j.anihpc.2015.07.001

(P) Find Ω with Lipschitz boundary such that $B \subseteq \Omega \subseteq D$ and which minimizes the *shape functional*

$$J(\Omega) := \min_{\substack{u \in W^{1,p}(\Omega) \\ u=g \text{ on } B}} \left[\int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} \beta(x)|u|^p \, d\mathcal{H}^{d-1} + \gamma|\Omega| \right],$$

where $p > 1$, $\gamma \geq 0$, $f : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty[$ is continuous, with $\xi \mapsto f(x, \xi)$ convex and positively p-homogeneous, and $\beta : \mathbb{R}^d \to [0, +\infty[$ is continuous ($\mathcal{H}^{d-1}$ stands for the $(d-1)$-dimensional Hausdorff measure).

The problem amounts to the determination of the "*free boundary*" $\partial\Omega$ of the optimal domain Ω on which the associated state function u (which realizes $J(\Omega)$) satisfies a boundary condition of *Robin type*. In the case $p = 2$, $f(x, \xi) = |\xi|^2$ and $\beta(x) = \beta$, the condition reduces precisely to the classical Robin condition

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on } \partial\Omega,$$

where n denotes the unit external normal.

The function u satisfies also extra conditions on $\partial\Omega$ coming from optimality. Those new conditions, are referred to be *overdetermined*, but they do not play a fundamental role in our approach to the minimization problem.

In the two dimensional case with $f(x, \xi) = A(x)\xi \cdot \xi$, $\gamma = 0$ and $p = 2$, the problem can be interpreted as that of finding the shape of the membrane with minimal total energy among those with elastic properties given by the elastic moduli $A(x)$, prescribed transversal displacement g on the part B, which are elastically supported at the boundary (with elastic forces with constant $\beta(x)$).

The existence of optimal domains for problem (P) is unclear. In general, there are very few results in shape optimization where the existence of an optimal domain can be proved in a "natural" way, i.e. without imposing extra restrictive conditions, and the most of them hold for Dirichlet boundary conditions. For Robin b.c., the only analysis carried to understand existence concerns the first eigenvalue of the Robin Laplacian. Contrary to Dirichlet b.c., the general relaxed form of a Robin problem (i.e., a precise description of the limit of a sequence of Robin problems on a sequence of *arbitrary, non-smooth, non-uniform* domains) is not known. In this paper, we analyze a class of energy type functionals generalizing the Bernoulli free boundary problem in a nonlinear framework complemented by elastic boundary conditions.

The main result of our paper consists in the identification of a class of admissible domains $\mathcal{A}_B(D)$ containing the Lipschitz ones on which the minimization of J can be carried out and can be considered as a *relaxation* of the original problem.

The class $\mathcal{A}_B(D)$ and the extension of J to such a class is suggested by the study of the following free discontinuity functional

$$F(u) := \int_{\mathbb{R}^d} f(x, \nabla u)\, dx + \int_{J_u} \beta(x)[(u^+)^p + (u^-)^p]\, d\mathcal{H}^{d-1} + \gamma|\{u > 0\}| \tag{1.1}$$

on the set of functions

$$\mathcal{F}_{B,g}(D) := \{u \in SBV(\mathbb{R}^d) : supp(u) \subseteq \bar{D}, u \geq 0, u = g \text{ on } B\}.$$

Here *SBV* denotes the space of *special functions of bounded variation* introduced by De Giorgi and Ambrosio [9] to deal with image segmentation problems. The link between J and F is obtained easily noticing that if u is the state function of the regular domain Ω (which we can assume positive), then the extension of u to $\mathbb{R}^d$ by zero outside Ω yields an element $\tilde{u}$ of $\mathcal{F}_{B,g}(D)$ such that

$$F(\tilde{u}) = \int_\Omega f(x, \nabla u)\, dx + \int_{\partial\Omega} \beta(x)|u|^p \, d\mathcal{H}^{d-1} + \gamma|\Omega| = J(\Omega). \tag{1.2}$$

The surface energy in (1.1) is rather unusual, involving the *sum* of the p-power of the traces of u. Its form, among the many yielding equality (1.2), is suggested by lower semicontinuity issues for the functional F.

We expect that the minimization problem

$$\min_{u \in \mathcal{F}_{B,g}(D)} F(u) \tag{1.3}$$