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Sard property for the endpoint map on some Carnot groups

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Abstract

In Carnot–Carathéodory or sub-Riemannian geometry, one of the major open problems is whether the conclusions of Sard's theorem holds for the endpoint map, a canonical map from an infinite-dimensional path space to the underlying finite-dimensional manifold. The set of critical values for the endpoint map is also known as abnormal set, being the set of endpoints of abnormal extremals leaving the base point. We prove that a strong version of Sard's property holds for all step-2 Carnot groups and several other classes of Lie groups endowed with left-invariant distributions. Namely, we prove that the abnormal set lies in a proper analytic subvariety. In doing so we examine several characterizations of the abnormal set in the case of Lie groups.

MSC: 53C17; 22F50; 22E25; 14M17

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1. Introduction

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $V \subseteq \mathfrak{g}$ be a subspace. Following Gromov [12, Sec. 0.1], we shall call the pair (G, V) a *polarized group*. Carnot groups are examples of polarized groups where V is the first layer of their stratification. To any polarized group (G, V) one associates the endpoint map:

$$\begin{aligned} \mathrm{End} : L^2([0,1], V) &\to G \\ u &\mapsto \gamma_u(1), \end{aligned}$$

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where γ_u is the curve on G leaving from the origin $e \in G$ with derivative $(\mathrm{d}L_{\gamma_u(t)})_e u(t)$, where L_g denotes the left translation by g.

The abnormal set of (G, V) is the subset $\mathrm{Abn}(e) \subset G$ of all singular values of the endpoint map. Equivalently, $\mathrm{Abn}(e)$ is the union of all *abnormal curves* passing through the origin (see Section 2.3). If the abnormal set has measure 0, then (G, V) is said to satisfy the *Sard Property*. Proving the Sard Property in the general context of polarized manifolds is one of the major open problems in sub-Riemannian geometry, see the questions in [17, Sec. 10.2] and Problem III in [4]. In this paper, we will focus on the following stronger versions of Sard's property in the context of groups.

Definition 1.1 *(Algebraic and Analytic Sard Property).* We say that a polarized group (G, V) satisfies the *Algebraic* (respectively, *Analytic*) *Sard Property* if its abnormal set $\mathrm{Abn}(e)$ is contained in a proper real algebraic (respectively, analytic) subvariety of G.

Our main results are summarized by:

Theorem 1.2. *The following Carnot groups satisfy the Algebraic Sard Property:*

(1) *Carnot groups of step* 2*;*
(2) *The free-nilpotent group of rank* 3 *and step* 3*;*
(3) *The free-nilpotent group of rank* 2 *and step* 4*;*
(4) *The nilpotent part of the Iwasawa decomposition of any semisimple Lie group equipped with the distribution defined by the sum of the simple root spaces.*

The following polarized groups satisfy the Analytic Sard Property:

(5) *Split semisimple Lie groups equipped with the distribution given by the subspace of the Cartan decomposition with negative eigenvalue.*
(6) *Split semisimple Lie groups equipped with the distribution defined by the sum of the nonzero root spaces.*

Earlier work [16] allows us

(7) *compact semisimple Lie groups equipped with the distribution defined by the sum of the nonzero root spaces, (i.e., the orthogonal to the maximal torus relative to a bi-invariant metric).*

Case (1) will be proved reducing the problem to the case of a smooth map between finite-dimensional manifolds and applying the classical Sard Theorem to this map. The proof will crucially use the fact that in a Carnot group of step 2 each abnormal curve is contained in a proper subgroup. This latter property may fail for step 3, see Section 6.3. However, a similar strategy together with the notion of *abnormal varieties*, see (2.20), might yield a proof of Sard Property for general Carnot groups.

The proof of cases (2)–(6) is based on the observation that, if $\mathcal{X}$ is a family of contact vector fields (meaning infinitesimal symmetries of the distribution) vanishing at the identity, then for any horizontal curve γ leaving from the origin with control u we have

$$(R_{\gamma(1)})_* V + (L_{\gamma(1)})_* V + \mathcal{X}(\gamma(1)) \subset \mathrm{Im}(\mathrm{d}\,\mathrm{End}_u) \subset T_{\gamma(1)} G.$$

Therefore if $g \in G$ is such that

$$(R_g)_* V + (L_g)_* V + \mathcal{X}(g) = T_g G, \tag{1.3}$$

then g is not a singular value of the endpoint map. In fact, if (1.3) is describable as a non-trivial system of polynomial inequations for g, then (G, V) has the Algebraic Sard Property. Case (3) was already proved in [15] by using an equivalent technique.