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Ann. I. H. Poincaré – AN 33 (2016) 1667–1673

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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On fractional Laplacians – 2

Roberta Musina [a,*,1], Alexander I. Nazarov [b,c,2]

[a] *Dipartimento di Matematica ed Informatica, Università di Udine, via delle Scienze, 206 – 33100 Udine, Italy*
[b] *St. Petersburg Department of Steklov Institute, Fontanka, 27, St. Petersburg, 191023, Russia*
[c] *St. Petersburg State University, Universitetskii pr. 28, St. Petersburg, 198504, Russia*

Received 9 March 2015; accepted 18 August 2015

Available online 28 August 2015

Dedicated to Louis Nirenberg on the occasion of his 90th birthday

Abstract

For $s > -1$ we compare two natural types of fractional Laplacians $(-\Delta)^s$, namely, the "Navier" and the "Dirichlet" ones.

MSC: 47A63; 35A23

Keywords: Fractional Laplacians; Nonlocal differential operators; Sobolev spaces

1. Introduction

Recall that the Sobolev space $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_s^2 = \int\limits_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\mathcal{F}u(\xi)|^2 \, d\xi,$$

see for instance Section 2.3.3 of the monograph [8]. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) \, dx.$$

For arbitrary $s \in \mathbb{R}$ we define fractional Laplacian in $\mathbb{R}^n$ by the quadratic form

$$Q_s[u] = ((-\Delta)^s u, u) := \int\limits_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi,$$

* Corresponding author.
E-mail addresses: roberta.musina@uniud.it (R. Musina), al.il.nazarov@gmail.com (A.I. Nazarov).
1 Partially supported by Miur-PRIN 2009WRJ3W7-001 "Fenomeni di concentrazione e problemi di analisi geometrica".
2 Supported by RFBR grant 14-01-00534 and by St. Petersburg University grant 6.38.670.2013.

http://dx.doi.org/10.1016/j.anihpc.2015.08.001
0294-1449/

with domain

$$\mathrm{Dom}(Q_s) = \{u \in \mathcal{S}'(\mathbb{R}^n) \,:\, Q_s[u] < \infty\}.$$

Let Ω be a bounded and smooth domain in $\mathbb{R}^n$. We put

$$H^s(\Omega) = \left\{u\big|_\Omega \,:\, u \in H^s(\mathbb{R}^n)\right\},$$

see [8, Sec. 4.2.1] and the extension theorem in [8, Sec. 4.2.3].

Also we introduce the space

$$\widetilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^n) \,:\, \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

By Theorem 4.3.2/1 of [8], for $s - \frac{1}{2} \notin \mathbb{Z}$ this space coincides with $H_0^s(\Omega)$, that is the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^s(\Omega)$, while for $s - \frac{1}{2} \in \mathbb{Z}$ one has $\widetilde{H}^s(\Omega) \subsetneq H_0^s(\Omega)$. Moreover, $\mathcal{C}_0^\infty(\Omega)$ is dense in $\widetilde{H}^s(\Omega)$.

We introduce the "Dirichlet" fractional Laplacian in Ω (denoted by $(-\Delta_\Omega)_D^s$) as the restriction of $(-\Delta)^s$. The domain of its quadratic form is

$$\mathrm{Dom}(Q_{s,\Omega}^D) = \{u \in \mathrm{Dom}(Q_s) \,:\, \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

Also we define the "Navier" fractional Laplacian as s-th power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q_{s,\Omega}^N[u] = ((-\Delta_\Omega)_N^s u, u) := \sum\nolimits_j \lambda_j^s \cdot |(u, \varphi_j)|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω, respectively, and $\mathrm{Dom}(Q_{s,\Omega}^N)$ consists of distributions in Ω such that $Q_{s,\Omega}^N[u] < \infty$.

It is well known that for $s = 1$ these operators coincide: $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$. We emphasize that, in contrast to $(-\Delta_\Omega)_N^s$, the operator $(-\Delta_\Omega)_D^s$ is not the s-th power of the Dirichlet Laplacian for $s \neq 1$. In particular, $(-\Delta_\Omega)_D^{-s}$ is not inverse to $(-\Delta_\Omega)_D^s$.

The present paper is the natural evolution of [6], where we compared the operators $(-\Delta_\Omega)_D^s$ and $(-\Delta_\Omega)_N^s$ for $0 < s < 1$. In the first result we extend Theorem 2 of [6].

Theorem 1. *Let $s > -1$, $s \notin \mathbb{N}_0$. Then for $u \in \mathrm{Dom}(Q_{s,\Omega}^D)$, $u \not\equiv 0$, the following relations hold:*

$$Q_{s,\Omega}^N[u] > Q_{s,\Omega}^D[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{1}$$

$$Q_{s,\Omega}^N[u] < Q_{s,\Omega}^D[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{2}$$

Next, we take into account the role of dilations in $\mathbb{R}^n$. We denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω. If $\Omega' \in F(\Omega)$, then any $u \in \mathrm{Dom}(Q_{s,\Omega}^D)$ can be regarded as a function in $\mathrm{Dom}(Q_{s,\Omega'}^D)$, and the corresponding form $Q_{s,\Omega'}^D[u]$ does not change. In contrast, the form $Q_{s,\Omega'}^N[u]$ does depend on $\Omega' \supset \Omega$. However, roughly speaking, the difference between these quadratic forms disappears as $\Omega' \to \mathbb{R}^n$.

Theorem 2. *Let $s > -1$. Then for $u \in \mathrm{Dom}(Q_{s,\Omega}^D)$ the following facts hold:*

$$Q_{s,\Omega}^D[u] = \inf_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if} \quad 2k < s < 2k+1, \quad k \in \mathbb{N}_0; \tag{3}$$

$$Q_{s,\Omega}^D[u] = \sup_{\Omega' \in F(\Omega)} Q_{s,\Omega'}^N[u], \quad \text{if} \quad 2k-1 < s < 2k, \quad k \in \mathbb{N}_0. \tag{4}$$

For $-1 < s < 0$ we also obtain a pointwise comparison result reverse to the case $0 < s < 1$ (compare with [6, Theorem 1]).

Theorem 3. *Let $-1 < s < 0$, and let $f \in \mathrm{Dom}(Q_{s,\Omega}^D)$, $f \geq 0$ in the sense of distributions, $f \not\equiv 0$. Then the following relation holds:*

$$(-\Delta_\Omega)_N^s f < (-\Delta_\Omega)_D^s f. \tag{5}$$