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Global existence for solutions of the focusing wave equation with the compactness property

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Abstract

We prove that every solution of the focusing energy-critical wave equation with the compactness property is global. We also give similar results for supercritical wave and Schrödinger equations.

Keywords: Focusing wave equation; Dynamics; Compactness; Global existence

1. Introduction

In this note we consider solutions with the compactness property for (mainly) the energy-critical wave equation in dimension $N \in \{3, 4, 5\}$. This is the equation

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u, \quad t \in I,\ x \in \mathbb{R}^N \\ (u, \partial_t u)_{\restriction t=0} = \vec{u}_0 \in \dot{H}^1 \times L^2, \end{cases} \tag{1.1}$$

where I is an interval ($0 \in I$), u is real-valued, $\dot{H}^1 = \dot{H}^1(\mathbb{R}^N)$ and $L^2 = L^2(\mathbb{R}^N)$. For such solutions $(u, \partial_t u) \in C^0(I, \dot{H}^1 \times L^2)$, we denote the maximal interval of existence $(T_-(u), T_+(u)) = I_{\max}(u)$. We say that a solution has *the compactness property* if there exists $\lambda(t) > 0$, $x(t) \in \mathbb{R}^N$, $t \in I_{\max}(u)$ such that

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$$K = \left\{ \left(\lambda^{\frac{N-2}{2}}(t) u\big(t, \lambda(t)x + x(t)\big), \lambda^{\frac{N}{2}}(t)\partial_t u\big(t, \lambda(t)x + x(t)\big) \right) \; : \; t \in I_{\max}(\vec{u}_0) \right\} \tag{1.2}$$

is precompact in $\dot{H}^1 \times L^2$. Solutions with the compactness property have been extensively studied (see for instance [9,2,4,5] for equation (1.1)). The reason for these studies is that, if one considers solutions to (1.1) such that

$$\sup_{0<t<T_+(u)} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$$

and which do not scatter, there always exist $t_n \to T_+(u)$ such that, up to modulation, $(u, \partial_t u)(t_n)$ weakly converges to $(U(0), \partial_t U(0))$ in $\dot{H}^1 \times L^2$, where U is a solution with the compactness property. This is Proposition 1.10 in [6], which, as is noted there, is valid for a wide class of dispersive equations (see also [14] for nonlinear Schrödinger equation in high space dimension). This clearly shows the crucial role played by solutions with the compactness property in the study of bounded solutions of dispersive equations, with no a priori size restriction. Note also that in the case of (1.1), a much more precise result (Theorem 1 in [6]) is valid.

In [2] we showed that, in the radial case, up to scaling and sign change, the only solution of (1.1) with the compactness property is the "ground-state" $W(x) = \left(\frac{1}{1+|x|^2/N(N-2)}\right)^{\frac{N-2}{2}}$, which is the only non-zero radial solution in $\dot{H}^1(\mathbb{R}^N)$ (up to scaling and sign change) of the elliptic equation $\Delta u + |u|^{\frac{4}{N-2}} u = 0$. In the sequel, we will denote by Σ the set of non-zero solutions to this elliptic equation. In [6], Proposition 1.8, a), we showed (by a simple virial argument), that if u has the compactness property, then $T_-(u) = -\infty$, or $T_+(u) = +\infty$. In [9], the second and third authors showed that if $T_+(u) < \infty$ and u has the compactness property, then there exists $x_+ \in \mathbb{R}^N$ such that if $t \in I_{\max}$,

$$\operatorname{supp}(u(t), \partial_t u(t)) \subset \{|x - x_+| \leq |T_+(u) - t|\}$$

and

$$\lim_{t \to T_+(u)} \frac{\lambda(t)}{T_+(u) - t} = 0.$$

In particular, the self-similar blow-up, given by $\lambda(t) \approx T_+(u) - t$, is excluded.

In [6], Proposition 1.8 b), we showed that, if u has the compactness property, then there exist two sequences $\{t_n^{\pm}\}$ in $(T_-(u), T_+(u))$, with $\lim_{n \to \pm\infty} = T_{\pm}(u)$ and two elements $Q^{\pm}$ of Σ and a vector $\vec{\ell}$ with $|\vec{\ell}| < 1$, such that, up to modulation, $(u(t_n^{\pm}), \partial_t u(t_n^{\pm})) \to (Q_{\vec{\ell}}^{\pm}(0), \partial_t Q_{\vec{\ell}}^{\pm}(0))$ strongly in $\dot{H}^1 \times L^2$, where $Q_{\vec{\ell}}^{\pm}$ is the Lorentz transform of the solution $Q^{\pm}$, given by

$$Q_{\vec{\ell}}^{\pm} = Q^{\pm}\left(\left(-\frac{t}{\sqrt{1 - |\vec{\ell}|^2}} + \frac{1}{|\vec{\ell}|^2} \left(\frac{1}{\sqrt{1 - |\vec{\ell}|^2}} - 1 \right) \vec{\ell} \cdot x \right) \vec{\ell} + x \right),$$

so that $Q_{\vec{\ell}}^{\pm}(t, x) = Q_{\vec{\ell}}^{\pm}(0, x - t\vec{\ell})$, which are clearly solutions of (1.1) with the compactness property. In [5], we proved that the class of solutions with the compactness property is invariant under Lorentz transformation. In light of this result, we have (see [6,5]) the rigidity conjecture for solutions with the compactness property: 0 and $Q_{\vec{\ell}}$, $Q \in \Sigma$, $|\vec{\ell}| < 1$, are the only solutions of (1.1) with the compactness property. In [5] we proved this conjecture, under a nondegeneracy assumption on Q^+ (or Q^-).

The main result in this note is an extension of the non-existence of self-similar solution with the compactness property in [9]. Namely:

Theorem 1. *Let u be a solution of (1.1) with the compactness property. Then u is global.*

Theorem 1 is proved in Section 2. We will also prove a similar result for supercritical nonlinear wave equation (see Section 3) and supercritical Schrödinger equation (see Section 4).

In all the article, we let $\vec{u} = (u, \partial_t u)$.