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On the asymptotic behaviour of solutions of the stationary Navier–Stokes equations in dimension 3

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Abstract

In this paper, we address the problem of determining the asymptotic behaviour of the solutions of the incompressible stationary Navier–Stokes system in the full space, with a forcing term whose asymptotic behaviour at infinity is homogeneous of degree -3. We identify the asymptotic behaviour at infinity of the solution. We prove that it is homogeneous and that the leading term in the expansion at infinity uniquely solves the homogeneous Navier–Stokes equations with a forcing term which involves an additional Dirac mass. This also applies to the case of an exterior domain.

Keywords: Incompressible Navier–Stokes equations; Stationary; Asymptotic behaviour

1. Introduction

We consider the incompressible stationary Navier–Stokes equations with a forcing term in $\mathbb{R}^3$:

$$-\Delta U + (U\cdot\nabla)U + \nabla p = f, \quad \operatorname{div} U = 0 \text{ in } \mathbb{R}^3, \qquad \lim_{|x|\to\infty} U(x) = 0. \tag{1.1}$$

The forcing f is given and the unknowns are the velocity field U and the scalar pressure p. Clearly p is uniquely (up to a constant) determined by f and U. For this reason, by solution we mean only the velocity field U. In other words, throughout this paper a solution of (1.1) is a vector field U such that there exists some p such that (1.1) is satisfied.

The aim of this paper is to determine the asymptotic behaviour of the solutions at infinity under reasonable assumptions on the forcing f. Several authors investigated this problem.

In [1] the authors studied the existence and uniqueness of solutions under a smallness assumption in the critical space $L^{3,\infty}$. Moreover, that article found an explicit asymptotic behaviour of the solutions with a decay as $O(\frac{1}{|x|^2})$

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provided that $\mathbb{P}\Delta^{-1}f$ is bounded by $C/(1+|x|^2)$, where $\mathbb{P}$ is the Leray projector. More precisely they showed the following expansion for the solution:

$$U(x) = \mathbb{P}\Delta^{-1}f(x) + m(x) : \int_{\mathbb{R}^3} U \otimes U + O\left(\frac{\ln|x|}{|x|^3}\right) \qquad \text{as } |x| \to \infty \tag{1.2}$$

where $m(x)$ is an explicit function homogeneous of degree -2 and smooth outside 0. Observe that $\mathbb{P}\Delta^{-1}$ is a convolution operator whose kernel is a homogeneous function of degree -1. Therefore, if f is sufficiently decaying at infinity, the condition that $|\mathbb{P}\Delta^{-1}f| \leqslant C/(1+|x|^2)$ imposed in [1] holds true if and only if $\int_{\mathbb{R}^3} f = 0$.

But since all terms in the expansion (1.2) are $O(\frac{1}{|x|^2})$, it excludes all solutions which are homogeneous of degree -1. In particular it excludes the very important case of Landau solutions. The Landau solutions were introduced by Landau in [10] and they are given by the explicit formula

$$v_1^c(x) = 2\frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x|-x_1)^2}, \qquad v_2^c(x) = 2\frac{x_2(cx_1-|x|)}{|x|(c|x|-x_1)^2}, \qquad v_3^c(x) = 2\frac{x_3(cx_1-|x|)}{|x|(c|x|-x_1)^2}$$

with pressure

$$p(x) = 4\frac{cx_1-|x|}{|x|(c|x|-x_1)^2}.$$

They verify (1.1) with forcing $f = \beta\delta$ where

$$\beta = \frac{8\pi c}{3(c^2-1)}\left(2 + 6c^2 - 3c(c^2-1)\log\left(\frac{c+1}{c-1}\right)\right)$$

and δ is the Dirac mass in 0 (see [4]). It was even shown by Šverák [12] that all homogeneous solutions of (1.1) on $\mathbb{R}^3 \setminus \{0\}$ with vanishing forcing are the Landau solutions.

It appears then that the relevant asymptotic behaviour at infinity of the solutions of (1.1) with forcing sufficiently decaying at infinity should rather be of order $O(1/|x|)$. And indeed, it was shown in [11] that small solutions of the stationary incompressible Navier–Stokes equations in an exterior domain of $\mathbb{R}^3$ behave like $v(x) + o(1/|x|)$ where v is some unknown vector field homogeneous of degree -1. Moreover, Korolev and Šverák [9] observed that the asymptotic profile v must be a Landau solution. More precisely, they proved that if U is small and verifies (1.1) with $f = 0$ in the exterior of a ball with no boundary conditions required, then there exists a such that $U = v^a + o(1/|x|)$ as $|x| \to \infty$.

Let us also mention the work [5] where the authors study the stationary Navier–Stokes flow around a rotating body. They obtain again that the asymptotic behaviour of the solution is given by a Landau solution when the speed of rotation of the body is sufficiently small. In [8], the authors prove that the asymptotic behaviour as $|x| \to \infty$ of time-periodic solutions is also given by a Landau solution.

Since the relevant asymptotic behaviour at infinity is homogeneous of degree -1 and since the forcing corresponding to a velocity homogeneous of degree -1 is homogeneous of degree -3, it makes sense to study the asymptotic behaviour of the solutions of (1.1) with a forcing whose asymptotic behaviour at infinity is homogeneous of degree -3.

Let $\alpha \in (0,1)$ be fixed once and for all. We will assume in the rest of this paper that the forcing term is of the form

$$f = \phi f_0 + f_1 \tag{1.3}$$

where

- f_0 is homogeneous of degree -3, locally bounded on $\mathbb{R}^3 \setminus \{0\}$;
- we have that $|f_1(x)| \leqslant C/(1+|x|)^{3+\alpha}$ for some constant C;
- $\phi \in C^\infty(\mathbb{R}^3; [0,1])$ is a radial cut-off function such that $\phi(x) = 0$ for $|x| \leqslant 1/2$ and $\phi(x) = 1$ for $|x| \geqslant 1$.

The questions that we ask ourselves are the following. Under what additional hypothesis on f_0 and f_1 there exists a solution U of (1.1) such that $|U(x)| \leqslant C/|x|$ for some constant C? When such a solution exists, how does it behave at infinity? In short, we give the following answers. If such a U exists then necessarily