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ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

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Generic regularity of conservative solutions to a nonlinear wave equation

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Received 9 February 2015; received in revised form 4 August 2015; accepted 17 December 2015

Abstract

The paper is concerned with conservative solutions to the nonlinear wave equation $u_{tt} - c(u)\big(c(u)u_x\big)_x = 0$. For an open dense set of $\mathcal{C}^3$ initial data, we prove that the solution is piecewise smooth in the t–x plane, while the gradient u_x can blow up along finitely many characteristic curves. The analysis is based on a variable transformation introduced in [7], which reduces the equation to a semilinear system with smooth coefficients, followed by an application of Thom's transversality theorem.

Keywords: Nonlinear wave equations; Generic regularity; Singularity

1. Introduction

Consider the quasilinear second order wave equation

$$u_{tt} - c(u)\big(c(u)u_x\big)_x = 0, \qquad t \in [0, T], \quad x \in \mathbb{R}. \tag{1.1}$$

On the wave speed c we assume

(A) The map $c : \mathbb{R} \mapsto \mathbb{R}_+$ is smooth and uniformly positive. The quotient $c'(u)/c(u)$ is uniformly bounded. Moreover, the following generic condition is satisfied:

$$c'(u) = 0 \qquad \Longrightarrow \qquad c''(u) \neq 0. \tag{1.2}$$

Notice that, by (1.2), the derivative $c'(u)$ vanishes only at isolated points.

The analysis in [7,3] shows that, for any initial data

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), \tag{1.3}$$

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http://dx.doi.org/10.1016/j.anihpc.2015.12.004

with $u_0 \in H^1(\mathbb{R})$, $u_1 \in \mathbf{L}^2(\mathbb{R})$, the Cauchy problem admits a unique conservative solution $u = u(t,x)$, Hölder continuous in the t–x plane. We recall that conservative solutions satisfy an additional conservation law for the energy, so that the total energy

$$\mathcal{E}(t) \;=\; \frac{1}{2}\int [u_t^2 + c^2(u)u_x^2]\,dx$$

coincides with a constant for a.e. time t. A detailed construction of a global semigroup of these solutions, including more singular initial data, was carried out in [15].

In the present paper we study the structure of these solutions. Roughly speaking, we prove that, for generic smooth initial data (u_0, u_1), the solution is piecewise smooth. Its gradient u_x blows up along finitely many smooth curves in the t–x plane. Our main result is

Theorem 1. *Let the function $u \mapsto c(u)$ satisfy the assumptions* **(A)** *and let $T > 0$ be given. Then there exists an open dense set of initial data*

$$\mathcal{D} \;\subset\; \Big(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\Big) \times \Big(\mathcal{C}^2(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})\Big)$$

such that, for $(u_0, u_1) \in \mathcal{D}$, the conservative solution $u = u(t,x)$ of (1.1)–(1.3) is twice continuously differentiable in the complement of finitely many characteristic curves γ_i, within the domain $[0,T] \times \mathbb{R}$.

Here it is understood that $\mathcal{D}$ is open and dense w.r.t. the topology of the space

$$\mathcal{U} \;\doteq\; \Big(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\Big) \times \Big(\mathcal{C}^2(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})\Big) \tag{1.4}$$

with norm

$$\big\|(u_0, u_1)\big\|_{\mathcal{U}} \;\doteq\; \|u_0\|_{\mathcal{C}^3} + \|u_0\|_{H^1} + \|u_1\|_{\mathcal{C}^2} + \|u_1\|_{\mathbf{L}^2}\,.$$

As usual, the $\mathcal{C}^k$ norm of a function f is defined as

$$\|f\|_{\mathcal{C}^k} \;\doteq\; \sum_{j=0}^{k}\left(\sup_{x\in\mathbb{R}} |D^j f(x)|\right),$$

while

$$\|f\|_{H^1} \;\doteq\; \left(\int |f(x)|^2\,dx + \int |Df(x)|^2\,dx\right)^{1/2}.$$

The assumption **(A)** implies that $c(\cdot)$ is a Morse function. It plays a key role in our proof. We remark, however, that the conclusion of the theorem can hold also in some cases where **(A)** fails. For example, if c is constant, then the classical d'Alembert's formula shows that any solution with smooth initial data will remain smooth for all positive times.

For the scalar conservation law in one space dimension, a well known result by Schaeffer [17] shows that generic solutions are piecewise smooth, with finitely many shocks on any bounded domain in the t–x plane. A similar result was proved by Dafermos and Geng [8], for a special 2×2 Temple class system of conservation laws. It remains an outstanding open problem to understand whether generic solutions to more general 2×2 systems (such as the p-system of isentropic gas dynamics) remain piecewise smooth, with finitely many shock curves.

The proof in [17] relies on the Hopf–Lax representation formula, while the proof in [8] is based on the analysis of solutions along characteristics. In the present paper we take a quite different approach, based on the representation of solutions in terms of a semilinear system introduced in [7]. In essence, the analysis in [7] shows that, after a suitable change of variables, the quantities

$$w \;\doteq\; 2\arctan\big(u_t + c(u)u_x\big), \qquad\qquad z \;\doteq\; 2\arctan\big(u_t - c(u)u_x\big),$$

satisfy a semilinear system of equations, w.r.t. new independent variables X, Y. See (2.16)–(2.20) in Section 2 for details. Since this system has smooth coefficients, starting with smooth initial data one obtains a globally defined

Please cite this article in press as: A. Bressan, G. Chen, Generic regularity of conservative solutions to a nonlinear wave equation, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2015.12.004