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Transience and multifractal analysis ☆

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Abstract

We study dimension theory for dissipative dynamical systems, proving a conditional variational principle for the quotients of Birkhoff averages restricted to the recurrent part of the system. On the other hand, we show that when the whole system is considered (and not just its recurrent part) the conditional variational principle does not necessarily hold. Moreover, we exhibit an example of a topologically transitive map having discontinuous Lyapunov spectrum. The mechanism producing all these pathological features on the multifractal spectra is transience, that is, the non-recurrent part of the dynamics.

Keywords: Multifractal analysis; Ergodic theory; Lyapunov exponents

1. Introduction

The dimension theory of dynamical systems has received a great deal of attention over the last fifteen years. Multifractal analysis is a sub-area of dimension theory devoted to study the complexity of level sets of invariant local quantities. Typical examples of these quantities are Birkhoff averages, Lyapunov exponents, local entropies and pointwise dimension. Usually, the geometry of a level set is complicated and in order to quantify its size or complexity tools such as Hausdorff dimension or topological entropy are used. Thermodynamic formalism is, in most cases, the main technical device used in order to describe the various multifractal spectra. In this note we will be interested in multifractal analysis of Birkhoff averages and of quotients of Birkhoff averages. That is, given a dynamical system

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$T : X \to X$ and functions $\phi, \psi : X \to \mathbb{R}$, with $\psi(x) > 0$, we will be interested in the level sets determined by the quotient of Birkhoff averages of ϕ with ψ. Let

$$\alpha_m = \alpha_{m,\phi,\psi} := \inf \left\{ \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} : x \in X \right\} \text{ and} \tag{1}$$

$$\alpha_M = \alpha_{M,\phi,\psi} := \sup \left\{ \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} : x \in X \right\}. \tag{2}$$

For $\alpha \in [\alpha_m, \alpha_M]$ we define the level set of points having quotient of Birkhoff average equal to α by

$$J(\alpha) = J_{\phi,\psi}(\alpha) := \left\{ x \in X : \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} = \alpha \right\}. \tag{3}$$

Note that these sets induce the so called *multifractal decomposition* of the repeller,

$$X = \bigcup_{\alpha=\alpha_m}^{\alpha_M} J(\alpha) \cup J',$$

where J' is the *irregular set* defined by,

$$J' = J'_{\phi,\psi} := \left\{ x \in X : \text{ the limit } \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \phi(T^i x)}{\sum_{i=0}^{n-1} \psi(T^i x)} \text{ does not exist} \right\}.$$

The *multifractal spectrum* is the function that encodes this decomposition and it is defined by

$$b(\alpha) = b_{\phi,\psi}(\alpha) := \dim_H(J_{\phi,\psi}(\alpha)),$$

where $\dim_H$ denotes the Hausdorff dimension (see Section 2.3 or [11] for more details). Note that if $\psi \equiv 1$ then $b_{\phi,1}$ gives a multifractal decomposition of Birkhoff averages. If the set X is a compact interval, the dynamical system is uniformly expanding with finitely many piecewise monotone branches and the potentials ϕ and ψ are Hölder, it turns out that the map $\alpha \mapsto b_{\phi,\psi}(\alpha)$ is very well behaved. Indeed, both $\alpha_{m,\phi,\psi}$ and $\alpha_{M,\phi,\psi}$ are finite and the map $\alpha \mapsto b_{\phi,\psi}(\alpha)$ is real analytic (see the work of Barreira and Saussol [3]).

In the case where either $\phi = \log|T'|$ or $\psi = \log|T'|$ the map $\alpha \mapsto b_{\phi,\psi}(\alpha)$ can often be determined by looking at a Legendre or Fenchel transform of a suitable pressure function. In this case the results have been extended well beyond the uniformly hyperbolic setting, see [15,17,19,22,29,30,34,35,39,46]. However without the assumption of uniform hyperbolicity it is no longer always the case that $\alpha \mapsto b_{\phi,\psi}(\alpha)$ will be analytic as shown in [17,28,34,35,46].

For more general functions ϕ and ψ the relationship to the Legendre or Fenchel transforms of certain pressure functions no longer holds. However in [3] it is shown $\alpha \mapsto b_{\phi,\psi}(\alpha)$ can still be related to suitable pressure functions. Some of these results were extended by Iommi and Jordan [24] to the case of expanding full-branched interval maps, with countably many branches. However, as already mentioned, in this situation it is not always the case that the spectrum is real analytic. In [24] it is shown that there will be regions where the spectrum does vary analytically but the transitions between these regions may not be analytic or even continuous. In the situation where the map is non-uniformly expanding, for example the Manneville–Pomeau map, it was shown in [17,34,35,46] that the Lyapunov spectrum (equivalently the local dimension spectrum for the measure for maximal entropy) has a phase transition. In the general case the spectrum may be related to those studied in [24]. In this case it will not always be continuous, see Section 6 of [24]. The lack of uniform hyperbolicity of the dynamical system being the reason for the irregular behavior of the multifractal spectrum.

Another important result in the study of multifractal analysis are the so-called conditional variational principles. Indeed, it has been shown for a very large class of dynamical systems (not necessarily uniformly hyperbolic) and for a large class of potentials (not necessarily Hölder) that the following holds:

$$b_{\phi,\psi}(\alpha) = \sup \left\{ \frac{h(\mu)}{\int \log|F'| \, d\mu} : \frac{\int \phi \, d\mu}{\int \psi \, d\mu} = \alpha \text{ and } \mu \in \mathcal{M} \right\},$$

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