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ANIHPC:2781

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Ann. I. H. Poincaré – AN ●●● (●●●●) ●●●–●●●

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

On the growth of Sobolev norms of solutions of the fractional defocusing NLS equation on the circle

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Received 7 October 2015; received in revised form 6 February 2016; accepted 19 February 2016

Abstract

This paper is devoted to the study of large time bounds for the Sobolev norms of the solutions of the following fractional cubic Schrödinger equation on the torus:

$$i\partial_t u = |D|^\alpha u + |u|^2 u, \quad u(0,\cdot) = u_0,$$

where α is a real parameter. We show that, apart from the case $\alpha = 1$, which corresponds to a half-wave equation with no dispersive property at all, solutions of this equation grow at a polynomial rate at most. We also address the case of the cubic and quadratic half-wave equations.

MSC: 37K40; 35B45

Keywords: Hamiltonian systems; Fractional nonlinear Schrödinger equation; Nonlinear wave equation; Dispersive properties

1. Introduction

In the study of Hamiltonian partial differential equations, understanding the large time dynamics of solutions is an important issue. In usual cases, the conservation of the Hamiltonian along trajectories enables to control one Sobolev norm of the solution (in the so-called *energy space*), but when solutions are globally defined and regular, higher norms could grow despite the conservation laws, reflecting an energy transfer to high frequencies. Even for notorious equations, such as the nonlinear Schrödinger equation on manifolds, it is an old problem to know whether such instability occurs [3], and often still an open question.

Let us start for instance from the particular case of the defocusing Schrödinger equation on the torus of dimension one, with a cubic nonlinearity:

$$i\partial_t u = -\partial_x^2 u + |u|^2 u, \quad u(0,\cdot) = u_0. \tag{1}$$

Here, u is a function of time $t \in \mathbb{R}$ and of space variable $x \in \mathbb{T}$, and $u_0 \in H^1(\mathbb{T})$. Equation (1) is Hamiltonian, and because of the energy conservation, its trajectories are bounded is $H^1(\mathbb{T})$. But it is also well-known that (1) is

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http://dx.doi.org/10.1016/j.anihpc.2016.02.002
0294-1449/

integrable (see [14,27]), with conservation laws ensuring that if u_0 belongs to $H^s(\mathbb{T})$ for some $s \in \mathbb{N}\backslash\{0\}$, then the solution u remains bounded in H^s (this is even true for any real $s \geq 1$ [17]).

In order to track down large time instability for Hamiltonian systems, Majda, McLaughlin and Tabak [20] suggested to replace the Laplacian in (1) by a whole family of pseudo-differential operators: the operators $|D|^\rho$ (sometimes written as $(\sqrt{-\partial_x^2})^\rho$) for real ρ. Recall that if $w = \sum_{k\in\mathbb{Z}} w_k e^{ikx}$ is a function on the torus, then

$$|D|^\rho w = \sum_{k\in\mathbb{Z}} |k|^\rho w_k e^{ikx}.$$

So we consider the following fractional Schrödinger equation:

$$i\partial_t u = |D|^\alpha u + |u|^2 u, \quad u(0,\cdot) = u_0, \tag{2}$$

where α is any positive number. If $\alpha = 2$, we recognize the classical Schrödinger equation (1). In the case $\alpha = 1$, (2) is a non-dispersive equation, called the (defocusing) "half-wave" equation:

$$i\partial_t u = |D|u + |u|^2 u, \quad u(0,\cdot) = u_0. \tag{3}$$

This half-wave equation has been studied by Gérard and Grellier in [9]. In particular, they show that the dynamics of (3) is related to the behaviour of the solutions of a toy model equation, called the cubic Szegő equation:

$$i\partial_t u = \Pi_+(|u|^2 u), \quad u(0,\cdot) = u_0, \tag{4}$$

where $\Pi_+ := \mathbf{1}_{D\geq 0}$ is the projection onto nonnegative Fourier modes. In a more precise way, equation (4) appears to be the completely resonant system of (3).

All the equations (2) derive from the Hamiltonian $\mathcal{H}_\alpha(u) := \frac{1}{2}(|D|^\alpha u, u) + \frac{1}{4}\|u\|_{L^4}^4$ for the symplectic structure endowed by the form $\omega(u,v) = \Im m(u,v)$, where $(u,v) := \int_{\mathbb{T}} u\bar{v}$ denotes the standard inner product on $L^2(\mathbb{T})$. The functional $\mathcal{H}_\alpha$ is therefore conserved along trajectories. Gauge invariance as well as translation invariance also imply the existence of two other conservation laws for equation (2):

$$Q(u) := \frac{1}{2}\|u\|_{L^2}^2$$
$$M(u) := (Du, u), \quad \text{where } D := -i\partial_x,$$

i.e. the mass and the momentum respectively. Starting from these observations, it has been proved that for $\alpha = 1$, equation (3) admits a globally defined flow in H^s with $s \geq \frac{1}{2}$ (see [9]). In the case of the half-wave equation, the Brezis–Gallouët inequality [4] also ensures that H^s-norms of solutions grow at most like $e^{\exp B|t|}$, for some constant $B > 0$ depending on s and on the initial data.

The question of the large time instability of global solutions of (2) thus naturally arises: is it possible to find smooth initial data whose corresponding orbits are not bounded in some H^s space, or at least not polynomially bounded?[1]

The cubic Szegő equation discloses this kind of instability, as recently shown in [10,11]: for generic smooth initial data, the corresponding solution of the Szegő equation in H^s is polynomially unbounded, for any $s > \frac{1}{2}$. Therefore it is reasonable to think that the same statement should hold for the half-wave equation (3), though such a result seems far beyond our reach at the current stage of the theory. Nevertheless the theorem we prove in this paper gives an a priori bound for all solutions of the half-wave equation:

Theorem 1. *Let $u_0 \in \mathcal{C}^\infty(\mathbb{T})$, and $t \mapsto u(t)$ the solution of the half-wave equation (3) such that $u(0) = u_0$. Given any integer $n \geq 0$, we have*

$$\|u(t)\|_{H^{1+n}} \leq Ce^{B|t|^2}, \quad \forall t \in \mathbb{R}, \tag{5}$$

where B can be chosen as $B_n\|u_0\|_{H^{1/2}}^8$ with $B_n > 0$ depending only on n, and where C can be chosen to depend only on n and $\|u_0\|_{H^{1+n}}$.

[1] We say that a solution $t \mapsto u(t)$ is *polynomially bounded* in H^s if there are positive constants C and A (not depending on time) such that for all $t \in \mathbb{R}$, $\|u(t)\|_{H^s} \leq C(1+|t|)^A$.

Please cite this article in press as: J. Thirouin, On the growth of Sobolev norms of solutions of the fractional defocusing NLS equation on the circle, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2016.02.002