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On the kinetic energy profile of Hölder continuous Euler flows

Philip Isett [a,*,1], Sung-Jin Oh [b,2]

[a] *Department of Mathematics, MIT, Cambridge, MA, United States*
[b] *Department of Mathematics, UC Berkeley, Berkeley, CA, United States*

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Abstract

In [8], the first author proposed a strengthening of Onsager's conjecture on the failure of energy conservation for incompressible Euler flows with Hölder regularity not exceeding $1/3$. This stronger form of the conjecture implies that anomalous dissipation will fail for a generic Euler flow with regularity below the Onsager critical space $L_t^\infty B_{3,\infty}^{1/3}$ due to low regularity of the energy profile.

The present paper is the second in a series of two papers whose results may be viewed as first steps towards establishing the conjectured failure of energy regularity for generic solutions with Hölder exponent less than $1/5$. The main result of this paper shows that any non-negative function with compact support and Hölder regularity $1/2$ can be prescribed as the energy profile of an Euler flow in the class $C_{t,x}^{1/5-\epsilon}$. The exponent $1/2$ is sharp in view of a regularity result of Isett [8]. The proof employs an improved greedy algorithm scheme that builds upon that in Buckmaster–De Lellis–Székelyhidi [1].

Keywords: Convex integration; Incompressible Euler equations; Weak solutions; Conservation of energy

1. Introduction

The present work concerns the construction of Hölder continuous solutions to the incompressible Euler equations on $\mathbb{R} \times \mathbb{R}^3$ and on $\mathbb{R} \times \mathbb{T}^3$

$$\begin{aligned} \partial_t v^l + \partial_j (v^j v^l) + \partial^l p &= 0 \\ \partial_j v^j &= 0 \end{aligned} \tag{E}$$

with a prescribed, possibly rough energy profile. As we consider solutions with fractional regularity, what we mean by a solution to (E) is a continuous velocity field $v : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ that together satisfy (E) in the sense of distributions.

* Corresponding author.
E-mail addresses: isett@math.mit.edu (P. Isett), sjoh@math.berkeley.edu (S.-J. Oh).
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The main result of the paper is the following Theorem:

Theorem 1.1 *(Euler flows with prescribed energy profile). Let $\alpha < 1/5$, let $I \subseteq \mathbb{R}$ be a bounded open interval, and let $\bar{e}(t) \geq 0$ be any non-negative function with compact support in I which belongs to the class $\bar{e}(t) \in C_t^\gamma$ for some $\gamma > \frac{2\alpha}{1-\alpha}$. Then:*

1. *There exists a weak solution (v, p) to the incompressible Euler equations in the class $v \in C_{t,x}^\alpha(\mathbb{R} \times \mathbb{T}^3)$ with support contained in*
$$\operatorname{supp} v \cup \operatorname{supp} p \subseteq I \times \mathbb{T}^3$$
such that the energy profile of v is equal to $\int_{\mathbb{T}^3} |v|^2(t,x)dx = \bar{e}(t)$ for all $t \in \mathbb{R}$.
2. *Moreover, one may choose a one parameter family of solutions (v_A, p_A), $0 \leq A \leq 1$, with the above properties such that the energy profile of v_A is equal to $\int_{\mathbb{T}^3} |v_A|^2(t,x)dx = A\bar{e}(t)$ and such that $\|v_A\|_{C_{t,x}^\alpha} \to 0$ as $A \to 0$.*

The assumption that $e(t)$ is at least $2\alpha/(1-\alpha)$-Hölder is sharp in view of a regularity result in [8], which states that the energy profile of an Euler flow in the class $v \in L_t^\infty C_x^\alpha$, $0 < \alpha \leq 1/3$, belongs to the class $e(t) \in C_t^{2\alpha/(1-\alpha)}$. We remark that our arguments also allow one to achieve an energy profile that does not have compact support provided the norm $\|e\|_{C_t^\gamma} = \sup_t |e(t)| + \sup_t \sup_{|\Delta t| \neq 0} \frac{|e(t+\Delta t) - e(t)|}{|\Delta t|^\gamma}$ is finite. Furthermore, the proof of Theorem 1.1 extends easily to the nonperiodic setting (cf. Theorem 3.2 below).

Theorem 1.1 on solutions with prescribed rough energy profiles builds upon works [5,4,1], which exhibit solutions whose energy profiles can be any given smooth, strictly positive function on a closed interval $[0, T]$. These results show in particular that for any $\alpha < 1/5$ it is possible to construct solutions with $C_{t,x}^\alpha$ regularity whose energy profiles are strictly increasing or strictly decreasing (which we expect to be nongeneric solutions, as in Conjecture 1 of [6]). Theorem 1.1 improves on these results by obtaining sharp regularity for the energy profile, and by removing the restriction of having a strictly positive lower bound on the desired energy profile.

To achieve these improvements, we develop a more delicate greedy algorithm for choosing the energy increments at each stage of the iteration, and develop a sharper form of the Main Lemma in the iteration that allows us to execute this algorithm. A quadratic commutator estimate akin to the one used in the proof of energy conservation in [3,2] (as well as the proof of the $2\alpha/(1-\alpha)$-Hölder estimate for the energy profile in [8]) plays a key role in the proof. Our proof is also greatly simplified by the fact that we are able to achieve an exponential (rather than double-exponential) growth of frequencies in the iteration. This simplification is available thanks to improvements used to localize the construction in [6].

Our motivation for pursuing Theorem 1.1 extends from a strengthening of Onsager's conjecture proposed in [8], which states that a generic Euler flow in the class $C_t C_x^\alpha$, $0 < \alpha \leq 1/3$ will have an energy profile of the minimal regularity allowed by the equation. We refer to [6] for a thorough discussion.

2. The Main Lemma

In this section, we present the Main Lemma that is responsible for the proof of Theorem 1.1. The purpose of this lemma is to describe precisely the result of one step of the convex integration procedure. Theorem 1.1 follows from iteration of this Lemma as we will explain in Section 3.

We start by recalling the Euler–Reynolds system introduced in [5]. A vector field v^l, scalar field p and symmetric tensor field R^{jl} are said to be an Euler–Reynolds flow if (v, p, R) together satisfy the following PDE in the sense of distributions.

$$\begin{aligned} \partial_t v^l + \partial_j (v^j v^l) + \partial^l p &= \partial_j R^{jl} \\ \partial_j v^j &= 0 \end{aligned} \tag{1}$$

We will consider Euler–Reynolds flows on the domain $\mathbb{R} \times \mathcal{M}$ where $\mathcal{M}$ may be either a torus $\mathcal{M} = \mathbb{T}^3$ or $\mathcal{M} = \mathbb{R}^3$.

The Main Lemma of the iteration summarizes how, given an initial Euler–Reynolds flow (v, p, R) satisfying certain estimates, it is possible to perturb the velocity field and pressure to obtain a new Euler–Reynolds flow (v_1, p_1, R_1)

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