ARTICLE IN PRESS
ANIHPC:2790

Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN ••• (••••) •••–•••

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Long time behavior for a dissipative shallow water model

V. Sciacca [a], M.E. Schonbek [b,*], M. Sammartino [a]

[a] *Department of Mathematics, University of Palermo, 90123 Palermo, Italy*
[b] *Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA*

Received 10 April 2015; received in revised form 21 April 2016; accepted 13 May 2016

Abstract

We consider the two-dimensional shallow water model derived in [29], describing the motion of an incompressible fluid, confined in a shallow basin, with varying bottom topography. We construct the approximate inertial manifolds for the associated dynamical system and estimate its order. Finally, working in the whole space $\mathbb{R}^2$, under suitable conditions on the time dependent forcing term, we prove the L^2 asymptotic decay of the weak solutions.

Résumé

Nous considérons le modèle d'eau peu profonde à deux dimensions dérivé dans [29], décrivant le mouvement d'un fluide incompressible, confinèe dans un bassin peu profond, avec topographie du fond variable. Nous construisons des variétés inertielles approximatives pour le système dynamique associé et nous estimons son ordre. Finalement, pour le espace $\mathbb{R}^2$ avec des conditions appropriées pour la force, nous prouvons la L^2 décroissance asymptotique des solutions faibles.

Keywords: Inertial manifolds; Attractors; Time decay; Fourier splitting method; Incompressible viscous fluids; Navier–Stokes equations

1. Introduction

In [29], the authors derived the following shallow water model:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p + \eta\boldsymbol{u} = $$

$$= b^{-1}\nabla\cdot[b\nu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \mathbf{I}\nabla\cdot\boldsymbol{u})] + \mathbf{f}, \tag{1.1a}$$

$$\nabla\cdot(b\boldsymbol{u}) = 0, \tag{1.1b}$$

* Corresponding author.
E-mail addresses: vincenzo.sciacca@unipa.it (V. Sciacca), schonbek@math.ucsc.edu (M.E. Schonbek), marco@math.unipa.it (M. Sammartino).

http://dx.doi.org/10.1016/j.anihpc.2016.05.003

Please cite this article in press as: V. Sciacca et al., Long time behavior for a dissipative shallow water model, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2016.05.003

$$\boldsymbol{u}(\boldsymbol{x}, t=0) = \boldsymbol{u}_0, \tag{1.1c}$$

$$\boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \qquad \boldsymbol{x} \in \partial\Omega, \tag{1.1d}$$

$$\boldsymbol{\tau} \cdot (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) \cdot \boldsymbol{\nu} = -\beta \boldsymbol{u} \cdot \boldsymbol{\tau} \quad \boldsymbol{x} \in \partial\Omega. \tag{1.1e}$$

In the above system $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently regular boundary $\partial\Omega$ and $\boldsymbol{u}(\boldsymbol{x}, t)$ denotes the velocity of the fluid at $\boldsymbol{x} \in \Omega$ and at time t. The smooth function $b(\boldsymbol{x})$ describes the bottom topography and satisfies $0 < b_i \le b(\boldsymbol{x}) \le b_s$, $\nu(\boldsymbol{x})$ is the viscosity, $\eta(\boldsymbol{x})$ is a positive smooth bounded function defined in Ω representing the combined actions of the friction at the bottom and the wind pressure, $\mathbf{I}$ is the identity, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are respectively the unity tangent and normal vector to the boundary $\partial\Omega$, $\beta(z)$ is a regular function defined in $\partial\Omega$ giving the friction coefficient at the boundary, and $\boldsymbol{f}(\boldsymbol{x})$ is the force term which describes the wind stress.

System (1.1a)–(1.1e) was derived in [29] from a three-dimensional anisotropic eddy viscosity model of an incompressible fluid confined to a shallow basin with varying bottom topography. To obtain the shallow water model (1.1a)–(1.1e), the authors assumed that the depth of the basin is much smaller than the typical horizontal length, and the typical velocity of the fluid is much smaller than the velocity of the gravity waves. This last assumption is equivalent to consider the fluid motion on time scales much longer than the period of the gravity waves so that averaging on time suppresses gravity waves. The same assumptions had been used in [5] starting from the Euler equations to derive the so called lake equations. The system (1.1a)–(1.1e) is therefore a generalization of the lake equations [28,27] as the effects of the viscous stresses are taken into account. In [29] the well posedness of the model was also established. Given that in large scale flows the Reynolds number can reach values like 10^9 or higher, the problem of the vanishing viscosity limit for models of geophysical interest is considered to be relevant, see e.g. [24] and references therein; the zero viscosity of the system (1.1a)–(1.1e) was in fact addressed in [16], while the case of degenerate topography was considered in [3,17].

In this paper we construct approximate inertial manifolds whose order decreases exponentially with respect to the dimension of the manifold. We give the dependence of all the constants with respect to the corresponding physical parameters and in particular we give explicitly the order of the approximate inertial manifolds.

When $\Omega = \mathbb{R}^2$ we address the problem of the asymptotic decay of the solutions. Under suitable conditions on the forcing term and of the initial datum, we show that the energy norm of weak solution has non-uniform decay. A weak solution which satisfies a generalized energy inequality is constructed following [32,35,21]. Then using the Fourier splitting method [42,43,49] non-uniform L^2 decay is obtained.

Similar decay questions were originally proposed by Leray in [25,26] for the Navier–Stokes equations. The first proof for decay without a rate was given by Masuda in [32] and by Kato in [22] in the case of null force and strong solutions with small data. Schonbek [42,43], using the Fourier Splitting Method, obtained the algebraic rate of decay for weak solution with large data. See also [2,15,20,23,30,48].

The main technical difficulties in the application of the above mentioned theories to system (1.1a)–(1.1e) originate: first from the fact that the incompressibility condition (1.1b) is weighted with the bottom topography; second from the presence, in (1.1a), of a non-standard dissipative operator. Therefore, besides several technical difficulties with respect to the classical 2D Navier–Stokes system, here we had to derive the appropriate exponential dichotomy as well adapted Agmon and Brezis–Gallouet inequalities. The technical details are postponed to an Appendix. Concerning the time decay in $\mathbb{R}^2$, a modified energy inequality allows us to use a modified Fourier splitting method but, the presence of bottom topography, gives rise to more complicated terms that require ad hoc estimates.

The plan of the paper is the following. In the next section, after introducing the appropriate mathematical settings for the model equations, we prove the existence of the Approximate Inertial Manifolds (AIM) and, then give the thickness of the thin neighborhood in terms of the data.

In section 3.1 we give the preliminary results to establish the decay of the solutions. In section 3.2 we prove the non-uniform asymptotic decay of the L^2 norm of the weak solution.

2. Bounded domain: approximate inertial manifolds

The concept of inertial manifold was introduced in [12], as part of the theory of dissipative differential equations. An inertial manifold for a semigroup associated to a dissipative dynamical system, is a finite dimensional Lipschitz manifold which is positively invariant, and attracts all the orbits exponentially [38,44,46]. To prove the existence of the inertial manifold it is necessary that the so called *spectral gap* condition [46] is verified. Unfortunately, this

Please cite this article in press as: V. Sciacca et al., Long time behavior for a dissipative shallow water model, Ann. I. H. Poincaré – AN (2016), http://dx.doi.org/10.1016/j.anihpc.2016.05.003