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Everywhere differentiability of viscosity solutions to a class of Aronsson's equations

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Abstract

We show the everywhere differentiability of viscosity solutions to a class of Aronsson equations in $\mathbb{R}^n$ for $n \geq 2$, where the coefficient matrices A are assumed to be uniformly elliptic and $C^{1,1}$. Our result extends an earlier important theorem by Evans and Smart [18] who have studied the case $A = I_n$ which correspond to the ∞-Laplace equation. We also show that every point is a Lebesgue point for the gradient.

In the process of proving the results we improve some of the gradient estimates obtained for the infinity harmonic functions. The lack of suitable gradient estimates has been a major obstacle for solving the $C^{1,\alpha}$ problem in this setting, and we aim to take a step towards better understanding of this problem, too.

A key tool in our approach is to study the problem in a suitable intrinsic geometry induced by the coefficient matrix A. Heuristically, this corresponds to considering the question on a Riemannian manifold whose the metric is given by the matrix A.

Keywords: L^∞-variational problem; Absolute minimizer; Everywhere differentiability; Aronsson's equation

1. Introduction

For any open set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, we consider the Aronsson equation:

$$\mathcal{A}_H[u](x) := \langle D_x(H(x, Du(x))), D_p H(x, Du(x))\rangle = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

where the Hamiltonian H is given by $H(x, p) = \langle A(x)p, p\rangle$. We denote the set of all uniformly elliptic matrices A of order n by $\mathscr{A}(\Omega)$. Our main result is the following theorem.

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Theorem 1.1. *Assume $A \in \mathscr{A}(\Omega) \cap C^{1,1}(\Omega)$. Then any viscosity solution $u \in C(\overline{\Omega})$ to the Aronsson equation (1.1) is everywhere differentiable in Ω.*

In order to show the robustness of the methodology, following Evans–Smart [18] we also show that every point is a Lebesgue point for the gradient.

Observe that when A is the identity matrix of order n, the Aronsson equation (1.1) becomes the infinity Laplace equation:

$$\Delta_\infty u := \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0 \quad \text{in} \quad \Omega. \tag{1.2}$$

G. Aronsson [1–4] initiated the study of the infinity Laplace equation (1.2) by deriving it as the Euler–Lagrange equation, in the context of L^∞-variational problems, of absolute minimal Lipschitz extensions (AMLE) or equivalently absolute minimizers (AM) of

$$\inf\left\{ \operatorname*{esssup}_{x\in\Omega} |Du|^2 : \; u \in \mathrm{Lip}(\Omega) \right\}. \tag{1.3}$$

Employing the theory of viscosity solutions of elliptic equations, Jensen [20] has first proved the equivalence between AMLEs and viscosity solutions of (1.2), and the uniqueness of both AMLEs and infinity harmonic functions under the Dirichlet boundary condition. See [26] and [6] for alternative proofs. For further properties of infinity harmonic functions, we refer the readers to the paper by Crandall–Evans–Gariepy [13] and the survey articles by Aronsson–Crandall–Juutinen [7] and Crandall [12].

For L^∞-variational problems involving Hamiltonian functions $H = H(x, z, p) \in C^2(\Omega \times \mathbb{R} \times \mathbb{R}^n)$, Barron, Jensen and Wang [8] have proved that an absolute minimizer of

$$\mathscr{F}_\infty(u, \Omega) = \operatorname*{esssup}_{x\in\Omega} H(x, u(x), Du(x)) \tag{1.4}$$

is a viscosity solution of (1.1), provided the level sets of H are convex in p-variable. Recall that a Lipschitz function $u \in \mathrm{Lip}(\Omega)$ is an *absolute minimizer* for $\mathscr{F}_\infty$, if for every open subset $U \Subset \Omega$ and $v \in \mathrm{Lip}(U)$, with $v|_{\partial U} = u|_{\partial U}$, it holds

$$\mathscr{F}_\infty(u, U) \le \mathscr{F}_\infty(v, U).$$

See [15,5,21,22] for related works on both Aronsson's equations (1.1) and absolute minimizers of $\mathscr{F}_\infty$. Recently, Bjorland, Caffarelli and Figalli [9] (see also [11]) studied the infinity fractional Laplacian, that is, the L^∞-variational problems associated to non-local Hamiltonian functions.

The regularity for infinity harmonic functions (or viscosity solutions to (1.2)) has attracted great interest recently. When $n = 2$, Savin [27] has showed the interior C^1-regularity, and Evans–Savin [17] have established the interior $C^{1,\alpha}$-regularity. Wang and Yu [29] have established the C^1-boundary regularity and, moreover, they have also extended Savin's C^1-regularity to the Aronsson equation (1.1) for uniformly convex $H(p) \in C^2(\mathbb{R}^2)$ [28]. When $n \ge 3$, Evans and Smart [18,19] have established the interior everywhere differentiability of infinity harmonic functions, whereas Wang and Yu [29] have extended this to the boundary differentiability. For the inhomogeneous infinity Laplace equation, the everywhere differentiability has been shown by Lindgren in [23]. In this paper, we extend the Evans–Smart [18,19] differentiability result to cover also the case of the Aronsson equation (1.1) for $A \in \mathscr{A}(\Omega) \cap C^{1,1}(\Omega)$ and $n \ge 2$. The result is not merely a straightforward generalization of the known theory, since there are several new difficulties in running the arguments.

The Evans–Smart method heavily relies on a linear approximation property proved earlier by Crandall, Evans and Gariepy [13]. This result states that the difference quotient, corresponding to the differentiability condition, has a convergent subsequence. Then one needs to show the uniqueness of the limit to conclude the result. A linear approximation property also holds for the Aronsson equation (1.1); see Lemma 4.1 below. Lemma 4.1 was first proved by Yu [30] for some general Hamiltonian functions $H(x, p)$; later a proof was given in [22] when $H(x, p) = \langle A(x)p, p\rangle$ based on an intrinsic geometry induced by the coefficient A.

For showing the uniqueness of the aforementioned limit, we need to establish certain gradient estimates. The standard approach has been to study the ϵ-regularized equation. After introducing the coefficient matrix, the regularized