ARTICLE IN PRESS
ANIHPC:2766

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Ann. I. H. Poincaré – AN ••• (••••) •••–•••

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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L^2-contraction for shock waves of scalar viscous conservation laws

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Received 22 July 2014; received in revised form 17 September 2015; accepted 22 October 2015

Abstract

We consider the L^2-contraction up to a shift for viscous shocks of scalar viscous conservation laws with strictly convex fluxes in one space dimension. In the case of a flux which is a small perturbation of the quadratic Burgers flux, we show that any viscous shock induces a contraction in L^2, up to a shift. That is, the L^2 norm of the difference of any solution of the viscous conservation law, with an appropriate shift of the shock wave, does not increase in time. If, in addition, the difference between the initial value of the solution and the shock wave is also bounded in L^1, the L^2 norm of the difference converges at the optimal rate $t^{-1/4}$. Both results do not involve any smallness condition on the initial value, nor on the size of the shock. In this context of small perturbations of the quadratic Burgers flux, the result improves the Choi and Vasseur's result in [7]. However, we show that the L^2-contraction up to a shift does not hold for every convex flux. We construct a smooth strictly convex flux, for which the L^2-contraction does not hold any more even along any Lipschitz shift.

MSC: 35L65; 35L67; 35B35; 35B40

Keywords: Viscous conservation laws; Shock wave; Stability; Contraction; Relative entropy

1. Introduction and main results

This paper is devoted to the study of L^2-contraction properties, up to a shift, for viscous shock waves of scalar viscous conservation laws with smooth strictly convex fluxes A in one space dimension:

$$\begin{aligned} &\partial_t U + \partial_x A(U) = \partial^2_{xx} U, \quad t > 0,\ x \in \mathbb{R}, \\ &U(0,x) = U_0(x). \end{aligned} \tag{1.1}$$

For any smooth strictly convex flux A, and any $u_-, u_+ \in \mathbb{R}$ with $u_- > u_+$, there exists a smooth function S_1 defined on $\mathbb{R}$, and $\sigma \in \mathbb{R}$, such that $S_1(x - \sigma t)$ is a traveling wave solution of Equation (1.1), connecting u_- at $-\infty$ to u_+ at $+\infty$. The function S_1 satisfies

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http://dx.doi.org/10.1016/j.anihpc.2015.10.004
0294-1449/© 2015 Published by Elsevier Masson SAS.

Please cite this article in press as: M.-J. Kang, A.F. Vasseur, L^2-contraction for shock waves of scalar viscous conservation laws, Ann. I. H. Poincaré – AN (2015), http://dx.doi.org/10.1016/j.anihpc.2015.10.004

$$-\sigma S_1'(\xi) + A(S_1)'(\xi) = S_1''(\xi),$$
$$\lim_{\xi\to\pm\infty} S_1 = u_\pm, \quad \lim_{\xi\to\pm\infty} S_1' = 0, \tag{1.2}$$

where σ is the speed of the shock determined by the Rankine–Hugoniot condition:

$$\sigma = \frac{A(u_+) - A(u_-)}{u_+ - u_-}. \tag{1.3}$$

Integrating (1.2), we find

$$-\sigma(S_1 - u_\pm) + A(S_1) - A(u_\pm) = S_1', \quad \lim_{\xi\to\pm\infty} S_1 = u_\pm. \tag{1.4}$$

There have been extensive studies on the stability of shock profiles of viscous conservation laws. When initial data U_0 is a small perturbation from the viscous shock S_1, the stability estimates have been shown in various way, such as the maximum principle, Evans function theory and the weighted norm approach based on the semigroup framework. This kind of results in the scalar case have been obtained by Goodman [12], Hopf [13], Howard [14], Nishihara [25], but also in the system case by Liu [21,22] and Zumbrun [23], Szepessy and Xin [29] (see also [11]). On the other hand, Freistühler and Serre [10] have shown the L^1-stability of viscous shock waves, without smallness condition, by combining energy estimates, a lap-number argument and a specific geometric observation on attractor of steady states. Moreover, their stability result still holds for any L^p space, $1 \le p \le \infty$. This result was improved by Kenig and Merle [16], with the uniform convergence to the viscous shock with respect to initial datas. The contraction property of viscous scalar conservation laws with respect to Wasserstein distances, was studied by Bolley, Brenier, and Loeper in [5], and Carrillo, Francesco and Lattanzio in [6].

In this article, we use the relative entropy method to study contraction properties in L^2 for viscous shocks to scalar viscous conservation laws. This work follows a program initiated in [18,19,28,31,32] concerning the relative entropy method for the study on the stability of inviscid shocks for the scalar or system of conservation laws verifying a certain entropy condition. The relative entropy method has been used as an important tool in the study of asymptotic limits to conservation laws as well. For incompressible limits, see Bardos, Golse, Levermore [1,2], Lions and Masmoudi [20], Saint Raymond [26]. For the compressible limit, see Tzavaras [30] in the context of relaxation and [3,4,15,24,33] in the context of hydrodynamical limits.

Our first result is on the L^2-contraction up to a shift for viscous shocks of (1.1) with a strictly convex flux A which has a perturbed form of quadratic function as

$$A(x) = ax^2 + g(x), \quad a > 0, \tag{1.5}$$

where g is a C^2-function satisfying $\|g''\|_{L^\infty(\mathbb{R})} < \frac{2}{11}a$. The following result shows also a rate of convergence toward the shock waves as $t^{-1/4}$, as long as the initial perturbation $U_0 - S_1$ is also bounded in L^1. Notice that the decay rate $t^{-1/4}$ is the same rate as the heat equation. Moreover, our result does not need any assumption on the spatial decay of the initial data, in contrast with previous works (see for example [13,25]).

Theorem 1.1. *Assume the flux A as in (1.5). For any given $u_- > u_+$, let S_1 be the associated viscous layer of (1.1) with endpoints u_- and u_+. Then, for any solution U to (1.1) with initial data U_0 satisfying $U_0 - S_1 \in L^2(\mathbb{R})$, the following L^2-contraction holds:*

$$\|U(t,\cdot + X(t)) - S_1\|_{L^2(\mathbb{R})} \le \|U_0 - S_1\|_{L^2(\mathbb{R})}, \quad t > 0, \tag{1.6}$$

for the shift $X(t)$ satisfying

$$\dot{X}(t) = \sigma - \frac{2a + \|g''\|_{L^\infty(\mathbb{R})}}{2(u_- - u_+)} \int_{-\infty}^{\infty} (U(t, x + X(t)) - S_1(x)) S_1'(x)\,dx,$$
$$X(0) = 0. \tag{1.7}$$

Furthermore, if $U_0 - S_1 \in L^1 \cap L^2(\mathbb{R})$, we have the following estimate for all $t > 0$,

$$\|U(t,\cdot + X(t)) - S_1\|_{L^2(\mathbb{R})} \le \frac{C_0\|U_0 - S_1\|_{L^2(\mathbb{R})}}{C_0 + t^{1/4}\|U_0 - S_1\|_{L^2(\mathbb{R})}}, \tag{1.8}$$