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Finite time blowup for the parabolic–parabolic Keller–Segel system with critical diffusion

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Abstract

The present paper is concerned with the parabolic–parabolic Keller–Segel system

$$\partial_t u = \operatorname{div}\left(\nabla u^{q+1} - u\nabla v\right), \quad t > 0, \ x \in \Omega,$$
$$\partial_t v = \Delta v - \alpha v + u, \quad t > 0, \ x \in \Omega,$$
$$(u, v)(0) = (u_0, v_0) \geq 0, \quad x \in \Omega,$$

with degenerate critical diffusion $q = q_\star := (N-2)/N$ in space dimension $N \geq 3$, the underlying domain Ω being either $\Omega = \mathbb{R}^N$ or the open ball $\Omega = B_R(0)$ of $\mathbb{R}^N$ with suitable boundary conditions. It has remained open whether there exist solutions blowing up in finite time, the existence of such solutions being known for the parabolic–elliptic reduction with the second equation replaced by $0 = \Delta v - \alpha v + u$. Assuming that $N = 3, 4$ and $\alpha > 0$, we prove that radially symmetric solutions with negative initial energy blow up in finite time in $\Omega = \mathbb{R}^N$ and in $\Omega = B_R(0)$ under mixed Neumann–Dirichlet boundary conditions. Moreover, if $\Omega = B_R(0)$ and Neumann boundary conditions are imposed on both u and v, we show the existence of a positive constant C depending only on N, Ω, and the mass of u_0 such that radially symmetric solutions blow up in finite time if the initial energy does not exceed $-C$. The criterion for finite time blowup is satisfied by a large class of initial data.

MSC: 35B44; 35B33; 35K15; 35K65; 35Q92

Keywords: Blowup; Radial symmetry; Nonlinear diffusion; Parabolic–parabolic Keller–Segel system

1. Introduction

This paper is concerned with the generalized parabolic–parabolic Keller–Segel system

$$\partial_t u = \operatorname{div}\left(\nabla u^{q+1} - u\nabla v\right), \quad t > 0, \ x \in \Omega, \tag{1.1}$$

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$$\tau \partial_t v = \Delta v - \alpha v + u \,, \quad t > 0 \,, \ x \in \Omega \,, \tag{1.2}$$

$$(u, v)(0) = (u_0, v_0) \geq 0 \,, \quad x \in \Omega \,, \tag{1.3}$$

where τ is a positive constant, and q and α are non-negative parameters. When Ω is a bounded domain of $\mathbb{R}^N$, the system (1.1)–(1.3) is supplemented with either Neumann boundary conditions

$$\partial_\nu u^{q+1} = \partial_\nu v = 0 \,, \quad t > 0 \,, \ x \in \partial\Omega \,, \tag{1.4}$$

or mixed Neumann–Dirichlet boundary conditions

$$\partial_\nu u^{q+1} - u \partial_\nu v = v = 0 \,, \quad t > 0 \,, \ x \in \partial\Omega \,. \tag{1.5}$$

A salient feature of non-negative solutions to (1.1)–(1.3) satisfying the boundary conditions (1.4) or (1.5) or a suitable decay condition at spatial infinity is the conservation of mass of u throughout time evolution, that is,

$$\|u(t)\|_1 = \|u_0\|_1 \qquad \text{for } t \in (0, T_{\max}) \,, \tag{1.6}$$

where $T_{\max}$ is the maximal existence time of the solution and $\|\cdot\|_p$ denotes the L^p-norm for $p \in [1, \infty]$. It is simply obtained by integrating (1.1) over the domain and using Green's formula, the boundary terms vanishing as a consequence of the boundary behavior. Another noteworthy property of (1.1)–(1.3) is that the energy $F[u, v]$ given by

$$F[u, v] := \int_\Omega \left(\frac{u^{q+1}}{q} + \frac{|\nabla v|^2}{2} + \frac{\alpha}{2} v^2 - uv \right) dx \,, \tag{1.7}$$

is a Liapunov functional, the term u^{q+1}/q being replaced by $u \ln u$ when $q = 0$.

The system (1.1)–(1.3) with $q = 0$ and $N = 2$ supplemented with Neumann boundary conditions (1.4) in a bounded domain was originally derived by Keller and Segel [17] as a model of aggregation of cells moving towards higher concentration gradients of a chemical substance generated by the cells. From a mathematical viewpoint, the aggregation of cells is defined as the blowup of $\|u(t)\|_\infty$ in finite time, that is,

$$\limsup_{t \to T_{\max}} \|u(t)\|_\infty = \infty \quad \text{for some finite} \quad T_{\max} \in (0, \infty) \,.$$

Since it was too difficult to treat the blowup issue in the parabolic–parabolic system, a simplified version with $\tau = 0$ was introduced. Its generalized form reads

$$\partial_t u = \operatorname{div} \left(\nabla u^{q+1} - u \nabla v \right) \,, \quad t > 0 \,, \ x \in \Omega \,, \tag{1.8}$$

$$0 = \Delta v - \alpha v + u \,, \quad t > 0 \,, \ x \in \Omega \,, \tag{1.9}$$

$$u(0) = u_0 \geq 0 \,, \quad x \in \Omega \,, \tag{1.10}$$

supplemented with the boundary conditions (1.4) or (1.5) when Ω is a bounded domain of $\mathbb{R}^N$. The system (1.8)–(1.10) is now a parabolic–elliptic system and can actually be reduced to a single nonlocal parabolic equation by expressing v in terms of u with the help of the Green function associated to the Laplace operator. This particular feature marks a serious difference between the "parabolic–parabolic" and "parabolic–elliptic" versions of the Keller–Segel system from a mathematical point of view.

These two systems have attracted considerable interest since they not only reproduce qualitatively some observed biological phenomena such as the aggregation of cells (also referred to as chemotactic collapse in the literature) but also display a wide variety of dynamical behaviors. Indeed, it is by now well-known that, when $N \geq 2$, there is a critical value $q_\star := (N - 2)/N$ of the parameter q which separates two different behaviors: when $q > q_\star$, the diffusion term $\operatorname{div}\left(\nabla u^{q+1}\right)$ dominates the attractive drift term $-\operatorname{div}(u \nabla v)$ and the initial value problems (1.1)–(1.3) and (1.8)–(1.10) are globally well-posed for a large class of integrable and non-negative initial data. When $q < q_\star$, the dynamics is rather governed by the attractive drift term leading to unbounded solutions, the diffusion term still allowing for the existence of global solutions for sufficiently small initial data. The critical case $q = q_\star$ offers an interesting novelty as a new parameter, the mass $\|u_0\|_1$ of the initial condition, comes into play: there is a threshold value $M_c(N)$ of this parameter (with $M_c(2) = 8\pi$) below which solutions exist globally and above which finite time

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