ARTICLE IN PRESS
YACHA:1129

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Wavelet characterizations of the atomic Hardy space H^1 on spaces of homogeneous type

Xing Fu, Dachun Yang *

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

A R T I C L E   I N F O

Article history:
Received 15 September 2015
Received in revised form 25 February 2016
Accepted 1 April 2016
Available online xxxx
Communicated by Pierre G. Lemarié-Rieusset

MSC:
primary 42B30
secondary 42C40, 30L99

Keywords:
Metric measure space of homogeneous type
Hardy space
Regular wavelet
Spline function
Unconditional basis
Atom
Molecule

A B S T R A C T

Let $(\mathcal{X}, d, \mu)$ be a metric measure space of homogeneous type in the sense of R.R. Coifman and G. Weiss and $H^1_{\rm at}(\mathcal{X})$ be the atomic Hardy space. Via orthonormal bases of regular wavelets and spline functions recently constructed by P. Auscher and T. Hytönen, together with obtaining some crucial lower bounds for regular wavelets, the authors give an unconditional basis of $H^1_{\rm at}(\mathcal{X})$ and several equivalent characterizations of $H^1_{\rm at}(\mathcal{X})$ in terms of wavelets, which are proved useful.

1. Introduction

The real variable theory of Hardy spaces $H^p(\mathbb{R}^D)$ on the D-dimensional Euclidean space $\mathbb{R}^D$ plays essential roles in various fields of analysis such as harmonic analysis and partial differential equations; see, for example, [35,33,7,34]. Meyer [30] established the equivalent characterizations of $H^1(\mathbb{R}^D)$ via wavelets. Liu [27] obtained several equivalent characterizations of the weak Hardy space $H^{1,\infty}(\mathbb{R}^D)$ via wavelets. Wu [37] further gave a wavelet area integral characterization of the weighted Hardy space $H^p_\omega(\mathbb{R}^D)$ for

* Corresponding author.
E-mail addresses: xingfu@mail.bnu.edu.cn (X. Fu), dcyang@bnu.edu.cn (D. Yang).

http://dx.doi.org/10.1016/j.acha.2016.04.001
1063-5203/

$p \in (0, 1]$. Later, via the vector-valued Calderón–Zygmund theory, García-Cuerva and Martell [9] obtained a characterization of $H^p_\omega(\mathbb{R}^D)$ for $p \in (0, 1]$ in terms of wavelets without compact supports.

It is well known that many classical results of harmonic analysis over Euclidean spaces can be extended to spaces of homogeneous type in the sense of Coifman and Weiss [4,5], or to RD-spaces introduced by Han, Müller and Yang [16] (see also [15,39]).

Recall that a quasi-metric space $(\mathcal{X}, d)$ equipped with a non-negative measure μ is called a *space of homogeneous type* in the sense of Coifman and Weiss [4,5] if $(\mathcal{X}, d, \mu)$ satisfies the following *measure doubling condition*: there exists a positive constant $C_{(\mathcal{X})} \in [1, \infty)$ such that, for all balls $B(x, r) := \{y \in \mathcal{X} : \ d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \le C_{(\mathcal{X})}\mu(B(x, r)),$$

which further implies that there exists a positive constant $\widetilde{C}_{(\mathcal{X})}$ such that, for all $\lambda \in [1, \infty)$,

$$\mu(B(x, \lambda r)) \le \widetilde{C}_{(\mathcal{X})}\lambda^n \mu(B(x, r)), \tag{1.1}$$

where $n := \log_2 C_{(\mathcal{X})}$. Let

$$n_0 := \inf\{n \in (0, \infty) : \ n \text{ satisfies } (1.1)\}. \tag{1.2}$$

It is obvious that n_0 measures the dimension of $\mathcal{X}$ in some sense and $n_0 \le n$. Observe that (1.1) with n replaced by n_0 may not hold true.

A space of homogeneous type, $(\mathcal{X}, d, \mu)$, is called a *metric measure space of homogeneous type* in the sense of Coifman and Weiss if d is a metric.

Recall that an RD-*space* $(\mathcal{X}, d, \mu)$ is defined to be a space of homogeneous type satisfying the following additional *reverse doubling condition* (see [16]): there exist positive constants a_0, $\widehat{C}_{(\mathcal{X})} \in (1, \infty)$ such that, for all balls $B(x, r)$ with $x \in \mathcal{X}$ and $r \in (0, \operatorname{diam}(\mathcal{X})/a_0)$,

$$\mu(B(x, a_0 r)) \ge \widehat{C}_{(\mathcal{X})}\mu(B(x, r))$$

(see [39] for more equivalent characterizations of RD-spaces). Here and hereafter,

$$\operatorname{diam}(\mathcal{X}) := \sup\{d(x, y) : \ x, y \in \mathcal{X}\}.$$

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type. In [5], Coifman and Weiss introduced the atomic Hardy space $H^{p,q}_{\rm at}(\mathcal{X}, d, \mu)$ for all $p \in (0, 1]$ and $q \in [1, \infty] \cap (p, \infty]$ and showed that $H^{p,q}_{\rm at}(\mathcal{X}, d, \mu)$ is independent of the choice of q, which is hereafter simply denoted by $H^p_{\rm at}(\mathcal{X}, d, \mu)$, and that its dual space is the Lipschitz space $\operatorname{Lip}_{1/p-1}(\mathcal{X}, d, \mu)$ when $p \in (0, 1)$, or the space $\operatorname{BMO}(\mathcal{X}, d, \mu)$ of functions with bounded mean oscillations when $p = 1$.

Recall that Coifman and Weiss [5] introduced the following *measure distance* ρ which is defined by setting, for all $x, y \in \mathcal{X}$,

$$\rho(x, y) := \inf\left\{\mu\left(B_d\right) : \ B_d \text{ is a ball containing } x \text{ and } y\right\}, \tag{1.3}$$

where the infimum is taken over all balls in $(\mathcal{X}, d, \mu)$ containing x and y; see also [28]. It is well known that, although all balls defined by d satisfy the axioms of the complete system of neighborhoods in $\mathcal{X}$ [and hence induce a (separated) topology in $\mathcal{X}$], the balls B_d are not necessarily open with respect to the topology induced by the quasi-metric d. However, by [28, Theorem 2], we see that there exists a quasi-metric $\widetilde{d}$ such that $\widetilde{d}$ is *equivalent* to d, namely, there exists a positive constant C such that, for all $x, y \in \mathcal{X}$,