Appl. Comput. Harmon. Anal. 44 (2018) 89–122

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Convex Optimization approach to signals with fast varying instantaneous frequency

Matthieu Kowalski [a,b], Adrien Meynard [a], Hau-tieng Wu [c,*]

[a] Laboratoire des Signaux et Systèmes, Univ. Paris-Sud, CNRS, CentraleSupélec, France
[b] Parietal project-team, INRIA, Neurospin, CEA-Saclay, France
[c] Department of Mathematics, University of Toronto, Toronto, Ontario, Canada

ARTICLE INFO

Article history:
Received 17 March 2015
Received in revised form 12 January 2016
Accepted 17 March 2016
Available online 11 April 2016
Communicated by Patrick Flandrin

Keywords:
Time-frequency analysis
Convex optimization
FISTA
Instantaneous frequency
Chirp factor

ABSTRACT

Motivated by the limitation of analyzing oscillatory signals composed of multiple components with fast-varying instantaneous frequency, we approach the time-frequency analysis problem by optimization. Based on the proposed adaptive harmonic model, the time-frequency representation of a signal is obtained by directly minimizing a functional, which involves few properties an "ideal time-frequency representation" should satisfy, for example, the signal reconstruction and concentrative time-frequency representation. FISTA (Fast Iterative Shrinkage-Thresholding Algorithm) is applied to achieve an efficient numerical approximation of the functional. We coin the algorithm as *Time-frequency bY COnvex OptimizatioN* (Tycoon). The numerical results confirm the potential of the Tycoon algorithm.

1. Introduction

Extracting proper features from the collected dataset is the first step toward data analysis. Take an oscillatory signal as an example. We might ask how many oscillatory components are inside the signal, how fast each component oscillates, how strong each component is, etc. Traditionally, Fourier transform is commonly applied to answer these questions. However, it has been well known for a long time that when the signal is not composed of harmonic functions, then Fourier transform might not perform correctly. Specifically, when the signal satisfies $f(t) = \sum_{k=1}^{K} A_k(t)\cos(2\pi\phi_k(t))$, where $K \in \mathbb{N}$, $A_k(t) > 0$ and $\phi_k'(t) > 0$ but $A_k(t)$ and $\phi_k'(t)$ are not constants, the momentary behavior of the oscillation cannot be captured by the Fourier transform. A lot of efforts have been made in the past few decades to handle this problem. Time-frequency (TF) analysis based on different principals [22] has attracted a lot of attention in the field and many variations are available. Well known examples include short time Fourier transform (STFT), continuous wavelet transform (CWT), Wigner–Ville distribution (WVD), chirplet transform [40], S-transform [47], etc.

* Corresponding author.
E-mail addresses: matthieu.kowalski@lss.supelec.fr (M. Kowalski), hauwu@math.toronto.edu (H.-t. Wu).

http://dx.doi.org/10.1016/j.acha.2016.03.008
1063-5203/

While these methods are widely applied in many fields, they are well known to be limited, again, by the Heisenberg uncertainty principle or the mode mixing problem caused by the interference known as the Moire patterns [22]. To alleviate the shortage of these analyses, in the past decades several solutions were proposed. For example, the empirical mode decomposition (EMD) [31] was proposed to study the dynamics hidden inside an oscillatory signal; however, its mathematical foundation is still lacking at this moment and several numerical issues cannot be ignored. Variations of EMD, like [52,42,25,44,21], were proposed to improve EMD. The sparsity approach [29,27,28,48] and iterative convolution-filtering [37,30,12,13] are other algorithms proposed to capture the flavor of the EMD, which have solid mathematical support. The problem could also be discussed via other approaches, like the optimized window approach [45], nonstationary Gabor frame [3], ridge approach [45], iterative Blaschke factorization [14], the approximation theory approach [11], non-local mean approach [24] and time-varying autoregression and moving average approach [19], to name but a few. Among these approaches, the reassignment technique [34,2,8,1] and the synchrosqueezing transform (SST) [17,16,9] have attracted more and more attention in the past few years. The main motivation of the reassignment technique is to improve the resolution issue introduced by the Heisenberg principal – the STFT coefficients are reallocated in *both* frequency axis and time axis according to their local phase information, which leads to the reassignment technique. The same reassignment idea can be applied to a very general settings like Cohen's class, affine class, etc. [23]. SST is a special reassignment technique; in SST, the STFT or CWT coefficients are reassigned *only* on the frequency axis [17,16,9] so that the causality is preserved and hence a real time algorithm is possible [10]. The same idea could be applied to different TF representation; for example, the SST based on wave packet transform or S-transform is recently considered in [53,32].

By carefully examining these methods, we see that there are several requirements a time series analysis method for an oscillatory signal should satisfy. First, if the signal is composed of several oscillatory components with different frequencies, the method should be able to decompose them. Second, if the oscillatory component has time-varying frequency or amplitude, then how the frequency or amplitude change should be well approximated. Third, if any of the oscillatory component exists only over a finite period, the algorithm should provide a clear information about the starting point and ending point. Fourth, if we represent the oscillatory behavior in the TF plane, then the TF representation should be sharp enough and contain the necessary information. Fifth, the algorithm should be robust to noise. Sixth, the analysis should be adaptive to the signal we want to analyze. However, not every method could satisfy all these requirements. For example, due to the Heisenberg uncertainty principle, the TF representation of the STFT is blurred; the EMD is sensitive to noise and is incapable of handling the dynamics of the signal indicated in the third requirement. In addition to the above requirements, based on the problem we have interest, other features are needed from the TF analysis method, and some of them might not be easily fulfilled by the above approaches.

Among these methods, SST [17,16,9] and its variation [35,53,32,43] could simultaneously satisfy these requirements, but it still has limitations. While SST could analyze oscillatory signals of "slowly varying" instantaneous frequency (IF) well with solid mathematical supports, the window needs to be carefully chosen if we want to analyze signals with fast varying IF [36]. Precisely, the conditions $|A_k'(t)| \leq \epsilon\phi_k'(t)$ and $|\phi_k''(t)| \leq \epsilon\phi_k'(t)$ are essential if we want to study the model $f(t) = \sum_{k=1}^{K} A_k(t)\cos(2\pi\phi_k(t))$ by the current SST algorithm proposed in [17,16,9]. Note that these "needs" could be understood/modeled as some suitable constraints, and to analyze the signal and simultaneously fulfill the designed constraints, optimization is a natural approach. Thus, in this paper, based on previous works and the above requirements, we would consider an optimization approach to study the oscillatory signals, which not only satisfies the above requirements, but also captures other features. In particular, we focus on capturing the fast varying IF. In brief, based on the relationship among the oscillatory components, the reconstruction property and the sparsity requirement on the time-frequency representation, we suggest to evaluate the optimal TF representation, denoted as F, by optimizing the following functional

$$\mathcal{H}(F,G) := \int \left| \Re \int F(t,\omega)\mathrm{d}\omega - f(t) \right|^2 \mathrm{d}t$$