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Generalized tight p-frames and spectral bounds for Laplace-like operators

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ABSTRACT

We prove sharp upper bounds for sums of eigenvalues (and other spectral functionals) of Laplace-like operators, including bi-Laplacians and fractional Laplacians. We show that among linear images of a highly symmetric domain, our spectral functionals are maximal on the original domain. We exploit the symmetries of the domain, and the operator, avoiding the necessity of finding good test functions for variational problems. This is especially important for fractional Laplacians, since exact solutions are not even known on intervals, making it hard to find good test functions.

To achieve our goals we generalize tight p-fusion frames, to extract the best possible geometric results for domains with isometry groups admitting tight p-frames. Any such group generates a tight p-fusion frame via conjugations of a fixed projection matrix. We show that generalized tight p-frames can also be obtained by conjugations of an arbitrary rectangular matrix, with the frame constant depending on the singular values of the matrix.

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1. Introduction

The dependence of the Laplace eigenvalues on the shape of the domain was extensively studied by many authors. In particular, shapes that maximize/minimize spectral functionals are of great interest (see monograph [35] for a comprehensive overview). We find the extremizing domains for spectral functionals of a broad family of Laplace-like operators, ranging from bi-Laplacians (and higher order operators) to fractional Laplacians (and other non-local operators). The important common feature of all treated operators is the invariance under isometries of the Euclidean space. We derive bounds for sums of eigenvalues of such operators on linearly transformed highly symmetric domains in terms of the eigenvalues on the symmetric domain. We obtain sharp bounds based purely on the symmetries of the domain and the operator, regardless if the eigenvalues and eigenfunctions of the symmetric domain are known explicitly. See Sec-

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tion 1.1 for bounds involving commonly used operators. This work extends [44] and [45] to more general operators.

Our proofs are based on the theory of tight p-frames, recently studied by Bachoc, Ehler and Okoudjou [29,5]. In Appendix A, we generalize certain aspects of this theory, to find an exact method of evaluating quadratic forms on transformed domains. This part of the paper may be of independent interest.

As discovered by Bachoc and Ehler [5], existence of tight p-frames is strongly connected to the theory of invariant polynomials for irreducible representations of finite groups of symmetries (subgroups of orthogonal groups). We emphasize and exploit this connection even further in Appendix A to generalize p-fusion frames involving subspace projection matrices to arbitrary matrices. Note that groups of symmetries have been previously used to study spectral theory by Bérard and Besson [15,14], Gurarie [32], Hoshikawa and Urakawa [36], Laugesen and the author [45,44].

The frame constants for our generalizations are particularly hard to evaluate. We meet the challenge (in Appendix B) using a mixture of techniques spanning a variety of fields of mathematics. We use probabilistic arguments to establish two combinatorial formulas involving symmetric polynomials. We use algebra of symmetric functions on abstract alphabets (including alphabet manipulations) to relate our combinatorial identities. Finally, the cycle index of the symmetric group provides the simplest form of the generalized p-frame constants.

1.1. Main results

In this section we present the eigenvalue estimates for a few commonly used Laplace-like operators. Most of these results are special cases of the general Fourier multiplier eigenvalue bound, Theorem 2.1. We present these special cases because of the importance of the operators as well as the relative simplicity of the statements of these results. It is also worth noting that Hardy–Littlewood–Pólya majorization can be used to further generalize all the results for sums of eigenvalues to sums of any concave increasing function of eigenvalues (see [46, Appendix A]). Some particularly interesting generalizations involve products of eigenvalues, partial sums of the spectral zeta function, and traces of the heat kernels (cf. [46, Theorem 1.1]). Results for majorized sums can be added to virtually any bound in this paper. For example Theorem 2.1 implies Corollary 2.2 [via this procedure].

Our examples naturally split into plate-related higher order multipliers, and fractional probability related multipliers. The common theme is that we transform a domain using an invertible linear transformation on $\mathbb{R}^d$. Throughout this section T will denote the transformation as well as its defining matrix.

1.1.1. Clamped plate with tension

Consider the eigenvalue problem for the bi-Laplacian operator with tension and clamped boundary conditions

$$\Delta^2 u - \tau\Delta u = \Gamma u \qquad \text{on } \Omega,$$
$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

Its eigenvalues Γ_i correspond to frequencies of oscillations (energy levels) of a rigid plate Ω with clamped boundary. The constant τ corresponds to the ratio of the lateral tension to the flexural rigidity of the plate. Positive values of τ mean that the plate is under tension, while negative values indicate compression. Note that the Laplacian is a negative operator, hence the "−" sign in front of τ ensures that the higher the tension, the higher the eigenvalues. The eigenvalues Γ_i were extensively studied both theoretically (e.g. Ashbaugh–Benguria [1], Ashbaugh–Benguria–Laugesen [2], Kawohl–Levine–Velte [38], Nadirashvili [54], Payne [55], Szegő [61,62], Talenti [63]) and numerically (e.g. Fichera [30], Kuttler–Sigillito [40,41], Leissa [48],

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