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Disjoint sparsity for signal separation and applications to hybrid inverse problems in medical imaging ☆

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ABSTRACT

The main focus of this work is the reconstruction of the signals f and g_i, $i = 1, \ldots, N$, from the knowledge of their sums $h_i = f + g_i$, under the assumption that f and the g_i's can be sparsely represented with respect to two different dictionaries A_f and A_g. This generalizes the well-known "morphological component analysis" to a multi-measurement setting. The main result of the paper states that f and the g_i's can be uniquely and stably reconstructed by finding sparse representations of h_i for every i with respect to the concatenated dictionary $[A_f, A_g]$, provided that enough incoherent measurements g_i are available. The incoherence is measured in terms of their mutual disjoint sparsity.

This method finds applications in the reconstruction procedures of several hybrid imaging inverse problems, where internal data are measured. These measurements usually consist of the main unknown multiplied by other unknown quantities, and so the disjoint sparsity approach can be directly applied. As an example, we show how to apply the method to the reconstruction in quantitative photoacoustic tomography, also in the case when the Grüneisen parameter, the optical absorption and the diffusion coefficient are all unknown.

1. Introduction

Hybrid, or coupled-physics, inverse problems have been extensively studied over the last years, both from the mathematical and the experimental points of view. A hybrid imaging modality consists in the combination of two types of techniques, one exhibiting the high contrast of tissues and a second one providing

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high resolution. Thus, the main drawbacks of the standard imaging modalities can be overcome, at least theoretically. Many combinations have been considered, such as optical and acoustic waves (photoacoustic tomography [54]), electric currents and ultrasounds (ultrasound modulated EIT [8]) or microwaves and ultrasounds (thermoacoustic tomography [54]). The reader is referred to [7,53,16,5,6,67] for a review on the mathematical aspects related to hybrid imaging problems.

In general, the inversion for such problems involves two steps. In the first step, an inverse problem related to the high resolution wave provides certain internal measurements. Such internal data are usually functionals of the unknown parameters and of certain solutions of partial differential equations (the unknowns are normally the coefficients of the PDE). In the second step, the unknown parameter has to be reconstructed from the knowledge of the internal measurements. This is sometimes referred to as the quantitative step, since the information on the tissue properties contained in the internal data is only qualitative. In this paper we suppose that the first inversion has been performed, and focus only on the second step.

The quantitative step is normally solved with PDE-based methods, by combining the internal data with the PDE modeling the problem. Such approach is sometimes very powerful in the reconstruction [21,13,16,2,3]. However, there may be difficulties in using these methods. First, the PDE model may be accurate only in some circumstances but not in others [16]. Second, even if the PDE model is accurate, there may be too many unknowns to have unique reconstruction [19]. Third, even in cases when the reconstruction is unique, this may require the differentiation of the data [13,14,1], which is known to be an unstable process, or may require additional assumptions to be satisfied [13,1,17,14,4].

The main focus of this paper is an alternative approach to such problem based on the use of sparse representations, as it was first done by Rosenthal et al. in quantitative photoacoustic tomography (QPAT) [64]. The internal data in a domain Ω can be often expressed as the product of the unknown(s) and an expression involving the solutions of the PDE. (For example, in QPAT the internal data have the form $H = \Gamma\mu u$, where Γ is the Grüneisen parameter, μ is the optical absorption and u is the light intensity.) Taking the logarithm, the inversion corresponds to recovering two functions f and g from the knowledge of their sum

$$h(x) = f(x) + g(x), \qquad x \in \Omega.$$

This problem is, in general, clearly unsolvable. However, it is possible to exploit the different levels of smoothness of f and g. Indeed, since f represents a property of the medium, such as the log conductivity, it is typically highly discontinuous. On the other hand, g is an expression involving the solutions of a PDE, and as such enjoys higher regularity properties. As a consequence, f and g have different features, and this can be used to separate them by using a sparse representation approach.

Two signals $f, g \in \mathbb{R}^n$ can be reconstructed from the knowledge of their sum $h = f + g$ provided that they have different characteristics. More precisely, they need to be sparsely represented, i.e., with few atoms, with respect to two incoherent dictionaries A_f and A_g. This method is usually called "morphological component analysis" (MCA), and was introduced by Starck et al. in [69] (see [25,45,44,50,24,23,42,56,71,60] for related works and [55] for a survey on the topic). In particular, Donoho and Kutyniok [42,56] first provided a theoretical foundation of geometric image separation into point and curve singularities by using tools from sparsity methodologies. If compared to these works, the novelty of this paper lies in the particular structure of the measurements, as we now describe.

In this work, motivated by hybrid imaging techniques, where multiple measurements with the parameters fixed can be taken, we extend this method to a multi-measurement setting. In general terms, this corresponds to the reconstruction of f (and g_i) from the knowledge of their sums

$$h_i = f + g_i, \qquad i = 1, \dots, N.$$

We prove that the MCA approach gives unique and stable reconstruction, provided that enough incoherent measurements g_i's are available. The incoherence is measured in terms of their mutual disjoint sparsity.