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Letter to the Editor

Projections and phase retrieval ☆

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ABSTRACT

We characterize collections of orthogonal projections for which it is possible to reconstruct a vector from the magnitudes of the corresponding projections. As a result we are able to show that in an M-dimensional real vector space a vector can be reconstructed from the magnitudes of its projections onto a generic collection of $N \geq 2M - 1$ subspaces. We also show that this bound is sharp when $N = 2^k + 1$. The results of this paper answer a number of questions raised in [5].

1. Introduction

The phase retrieval problem is an old one in mathematics and its applications. The author and his collaborators [2,6] previously considered the problem of reconstructing a vector from the magnitudes of its frame coefficients. In this paper we answer questions raised in the paper [5] about phase retrieval from the magnitudes of orthogonal projections onto a collection of subspaces.

To state our result we introduce some notation. Given a collection of proper linear subspaces $L_1, \ldots, L_N$ of $\mathbb{R}^M$ we denote by $P_1, \ldots, P_N$ the corresponding orthogonal projections onto the L_i. Assuming that the linear span of the L_i is all of $\mathbb{R}^M$ then any vector x can be recovered from vectors $P_1x, \ldots, P_Nx$ since the linear map

$$\mathbb{R}^M \to L_1 \times L_2 \times \ldots L_N, x \mapsto (P_1x, \ldots, P_Nx)$$

is injective.

When the P_i are all rank 1 then a choice of generator for each line determines a frame and the inner products $\langle P_ix, x\rangle$ are the frame coefficients with respect to this frame.

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In this paper we consider the problem, originally raised in [5], of reconstructing a vector x (up to a global sign) from the magnitudes

$$\|P_1 x\|, \|P_2 x\|, \dots, \|P_N x\|$$

of the projection vectors $P_1 x, \dots, P_N x$.

Let $\Phi = \{P_1, \dots, P_N\}$ be a collection of projections of ranks $k_1, \dots, k_N$. Define a map $\mathcal{A}_\Phi \colon (\mathbb{R}^M \setminus \{0\})/\pm 1 \to \mathbb{R}^N_{\geq 0}$ by the formula

$$x \mapsto (\langle P_1 x, P_1 x\rangle, \dots, \langle P_N x, P_N x\rangle)$$

As was the case for frames, injectivity of the map $\mathcal{A}_\Phi$ implies that phase retrieval by this collection of projections is theoretically possible.

In [5], Cahill, Casazza, Peterson and Woodland proved that there exist collections of $2M-1$ projections of rank more than one which allow phase retrieval. They also proved that a collection $\Phi = \{P_1, \dots, P_N\}$ of projections admits phase retrieval if and only if for every orthonormal basis $\{\phi_{i,d}\}_{d=1}^{k_d}$ of the linear subspace L_i determined by P_i the set of vectors $\{\phi_{i,d}\}_{i=1, d=1}^{N,\ k_d}$ allows phase retrieval.

Our first result is a more intrinsic characterization of collections of projections for which $\mathcal{A}_\Phi$ is injective.

Theorem 1.1. *The map $\mathcal{A}_\Phi$ is injective if and only if for every non-zero $x \in \mathbb{R}^M$ the vectors $P_1 x, \dots, P_N x$ span an M-dimensional subspace of $\mathbb{R}^M$, or equivalently the vectors $P_1 x, \dots, P_N x$ form an N-element frame in $\mathbb{R}^M$.*

As a corollary we obtain the following necessity result.

Corollary 1.2. *If $N \leq 2M-2$ and at least $M-1$ of the P_i have rank one, or if $N \leq 2M-3$ and at least $M-1$ of the P_i have rank $M-1$ then $\mathcal{A}_\Phi$ is not injective.*

Remark 1.3. We will see below that when the P_i all have rank one the condition of the theorem is equivalent to the corresponding frame having the finite complement property of [2].

Using the characterization of Theorem 1.1 we show that when $N \geq 2M-1$ any generic collection of projections admits phase retrieval. Note that this bound of $2M-1$ is the same as that obtained in [2].

Theorem 1.4. *If $N \geq 2M-1$, then for a generic collection $\Phi = (P_1, \dots, P_N)$ of ranks $k_1, \dots, k_N$ with $1 \leq k_i \leq M-1$, the map $\mathcal{A}_\Phi$ is injective.*

Remark 1.5. By generic we mean that Φ corresponds to a point in a non-empty Zariski open subset[1] of a product of real Grassmannians (which has the natural structure as an *affine variety*) whose complement has strictly smaller dimension. As noted in [3] one consequence of the generic condition is that for any continuous probability distribution on this variety, $\mathcal{A}_\Phi$ is injective with probability one. In particular Theorem 1.4 implies that phase retrieval can be done with $2M-1$ random subspaces of $\mathbb{R}^M$. This answers Problems 5.2 and 5.6 of [5]. We refer the reader to the paper of Bachoc and Ehler [1] for results on the feasibility of phase retrieval using collections of random linear projections.

In [2] it was proved that $N \geq 2M-1$ is a necessary condition for frames. Here, we obtain the following necessity result. This result was independently obtained by Zhiqiang Xu in his recent paper [12].

[1] See Section 2 for the definition of the Zariski topology.