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Gabor systems and almost periodic functions [☆]

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ABSTRACT

Inspired by results of Kim and Ron, given a Gabor frame in $L^2(\mathbb{R})$, we determine a non-countable generalized frame for the non-separable space $AP_2(\mathbb{R})$ of the Besicovic almost periodic functions. Gabor type frames for suitable separable subspaces of $AP_2(\mathbb{R})$ are constructed. We show furthermore that Bessel-type estimates hold for the AP norm with respect to a countable Gabor system using suitable almost periodic norms of sequences.

1. Introduction

Almost periodic functions on $\mathbb{R}$ are a natural generalization of usual periodic functions. Their first definition originates with the works of H. Bohr in 1924–1926. Since then a rich theory has been developed by a number of authors, particularly significant among others were the contributions by H. Weyl, N. Wiener, A.S. Besicovich, S. Bochner, V.V. Stepanov, C. de la Vallée Poussin and on LCA groups by J. Von Neumann.

Almost periodic functions found applications in various areas of harmonic analysis, number theory or group theory. In connection with dynamical systems and differential equations they were studied by B.M. Levitan and V.V. Zhikov [16] starting from results of J. Favard. In the more general context of

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pseudo-differential calculus M.A. Shubin [22] introduced scales of Sobolev spaces of almost periodic functions and recently A. Oliaro, L. Rodino and P. Wahlberg [17] extended some of these results to almost periodic Gevrey classes.

Another recent direction of research is connected with frame theory. As the main spaces of almost periodic functions are non-separable they cannot admit countable frames. The problem then arises in which sense frame-type inequalities are still possible for norm estimations in these spaces. Results in this direction were obtained by F. Galindo [8], and J.R. Partington/B. Unalmis [18,23], using windowed Fourier transform and wavelet transform, involving therefore Gabor and wavelet type frames. Further developments were recently obtained by Y.H. Kim and A. Ron [15,14], with fiberization techniques developed by A. Ron and Z. Shen [19,21] in the context of shift-invariant systems.

This paper has been inspired by the above-mentioned results of Kim–Ron. With different techniques we shall recover and extend some of their results concerning almost periodic norm estimates using Gabor systems. We postpone the description after we have introduced the definitions and properties we need. We refer e.g. to [2,4,5] for detailed presentations of the theory of almost periodic functions, and e.g. to [3,9,11] for frame theory.

The first type of functions we consider is, according to Bohr, the following.

Definition 1. The space $AP(\mathbb{R})$ of *almost periodic functions* is the set of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with the property that for every $\epsilon > 0$, there exists $L > 0$, such that every interval of the real line of length greater than L contains a value τ satisfying

$$\sup_t |f(t+\tau) - f(t)| \le \epsilon.$$

Each number τ, as described above, is called ϵ-period. We recall that $AP(\mathbb{R})$ coincides with the uniform norm closure of the space of trigonometric polynomials

$$\sum_{j=1}^{N} c_j e_{\lambda_j}$$

with $e_{\lambda_j}(t) = e^{i\lambda_j t}$ and $\lambda_j \in \mathbb{R},\ c_j \in \mathbb{C}$.

The norm $\|.\|_{AP(\mathbb{R})}$ associated with the inner product

$$(f,g)_{AP(\mathbb{R})} = \lim_{T\to+\infty} (2T)^{-1} \int_{-T}^{T} f(t)\overline{g(t)}\, dt$$

makes $AP(\mathbb{R})$ a non-complete, non-separable space.

Definition 2. The completion of $AP(\mathbb{R})$ with respect to this norm is the Hilbert space $AP_2(\mathbb{R})$ (sometimes also indicated as B^2) of *Besicovich almost periodic functions.*

$AP_2(\mathbb{R})$ also coincides with the completion of the trigonometric polynomials with respect to the norm $\|.\|_{AP(\mathbb{R})}$. The set $\{e_\lambda : \lambda \in \mathbb{R}\}$ is a non-countable orthonormal basis of $AP_2(\mathbb{R})$, consequently for every $f \in AP_2(\mathbb{R})$ the expansion $f = \sum_\lambda (f, e_\lambda)_{AP_2(\mathbb{R})}\, e_\lambda$ holds, where at most countably many terms are different form zero and the norm is given by Parseval's equality $\|f\|^2_{AP_2(\mathbb{R})} = \sum_{\lambda\in\mathbb{R}} \left|(f, e_\lambda)_{AP_2(\mathbb{R})}\right|^2$.

We will refer to $(f, e_\lambda)_{AP(\mathbb{R})}$ as the λ-th Fourier coefficient of $f \in AP(\mathbb{R})$ and from now on we will write $\widehat{f}(\lambda) = (f, e_\lambda)_{AP(\mathbb{R})}$. For a given function $f \in L^1(\mathbb{R})$ we also put