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Low rank matrix recovery from rank one measurements

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ABSTRACT

We study the recovery of Hermitian low rank matrices $X \in \mathbb{C}^{n \times n}$ from undersampled measurements via nuclear norm minimization. We consider the particular scenario where the measurements are Frobenius inner products with random rank-one matrices of the form $a_j a_j^*$ for some measurement vectors $a_1, \ldots, a_m$, i.e., the measurements are given by $b_j = \operatorname{tr}(X a_j a_j^*)$. The case where the matrix $X = xx^*$ to be recovered is of rank one reduces to the problem of phaseless estimation (from measurements $b_j = |\langle x, a_j \rangle|^2$) via the PhaseLift approach, which has been introduced recently. We derive bounds for the number m of measurements that guarantee successful uniform recovery of Hermitian rank r matrices, either for the vectors a_j, $j = 1, \ldots, m$, being chosen independently at random according to a standard Gaussian distribution, or a_j being sampled independently from an (approximate) complex projective t-design with $t = 4$. In the Gaussian case, we require $m \geq Crn$ measurements, while in the case of 4-designs we need $m \geq Crn \log(n)$. Our results are uniform in the sense that one random choice of the measurement vectors a_j guarantees recovery of all rank r-matrices simultaneously with high probability. Moreover, we prove robustness of recovery under perturbation of the measurements by noise. The result for approximate 4-designs generalizes and improves a recent bound on phase retrieval due to Gross, Krahmer and Kueng. In addition, it has applications in quantum state tomography. Our proofs employ the so-called bowling scheme which is based on recent ideas by Mendelson and Koltchinskii.

1. Introduction

1.1. The phase retrieval problem

The problem of retrieving a complex signal from measurements that are ignorant towards phases is abundant in many different areas of science, such as X-ray crystallography [44,62], astronomy [33] diffraction

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imaging [72,62] and more [8,12,81]. Mathematically formulated, the problem consists of recovering a complex signal (vector) $x \in \mathbb{C}^n$ from measurements of the form

$$|\langle a_j, x\rangle|^2 = b_j \quad \text{for} \quad j = 1, \ldots, m, \tag{1}$$

where $a_1, \ldots, a_m \in \mathbb{C}^n$ are sampling vectors. This ill-posed inverse problem is called *phase retrieval* and has attracted considerable interest over the last few decades. An important feature of this problem is that the signal x enters the measurement process (1) quadratically. This leads to a non-linear inverse problem. Classical approaches to numerically solving it include alternating projection methods [34,38]. However, these methods usually require extra constraints and a careful selection of parameters, and in particular, no rigorous convergence or recovery guarantees seem to be available.

As Balan et al. pointed out in [7] that this apparent obstacle of having nonlinear measurements can be overcome by noting that the measurement process – while quadratic in x – is linear in the outer product xx^*:

$$|\langle a_j, x\rangle|^2 = \operatorname{tr}\left(a_j a_j^* x x^*\right).$$

This "lifts" the problem to a matrix space of dimension n^2, where it becomes linear and can be solved explicitly, provided that the number of measurements m is at least n^2 [7]. However, there is additional structure present, namely the matrix $X = xx^*$ is guaranteed to have rank one. This connects the phase retrieval problem to the young but already extensive field of *low-rank matrix recovery*. Indeed, it is just a special case of low-rank matrix recovery, where both the signal $X = xx^*$ and the measurement matrices $A_j = a_j a_j^*$ are constrained to be proportional to rank-one projectors. This observation led to the *PhaseLift* approach to the phase retrieval problem [13,19].

It should be noted, however, that such a reduction to a low rank matrix recovery problem is just one possibility to retrieve phases. Other approaches use polarization identities [2] or alternate projections [65]. Another approach is quasi-linear compressed sensing [31]. Yet another recent method is phase retrieval via Wirtinger flow [15].

1.2. Low rank matrix recovery

Building on ideas of compressive sensing [20,30,37], low rank matrix recovery aims to reconstruct a matrix of low rank from incomplete linear measurements via efficient algorithms [68]. For our purposes we concentrate on Hermitian matrices $X \in \mathbb{C}^{n\times n}$ and consider measurements of the form

$$\operatorname{tr}(XA_j) = b_j \quad j = 1, \ldots, m \tag{2}$$

where the $A_j \in \mathbb{C}^{n\times n}$ are some Hermitian matrices. For notational simplicity, we define the measurement operator

$$\mathcal{A} : \mathcal{H}_n \to \mathbb{R}^m \quad Z \mapsto \sum_{j=1}^{m} \operatorname{tr}(ZA_j)\, e_j,$$

where $e_1, \ldots, e_m$ denotes the standard basis in $\mathbb{R}^m$. This summarizes an entire (possibly noisy) measurement process via

$$b = \mathcal{A}(X) + \epsilon. \tag{3}$$

Here $b = (b_1, \ldots, b_m)^T$ contains all measurement outcomes and $\epsilon \in \mathbb{R}^m$ denotes additive noise. Low rank matrix recovery can be regarded as a non-commutative version of compressive sensing. Indeed, the structural

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