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On the stability of sparse convolutions

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ABSTRACT

We give a stability result for sparse convolutions on $\ell^2(G) \times \ell^1(G)$ for torsion-free discrete Abelian groups G such as $\mathbb{Z}$. It turns out, that the torsion-free property prevents full cancellation in the convolution of sparse sequences and hence allows to establish stability, that is, injectivity with an universal lower norm bound, which only depends on the support cardinalities of the sequences. This can be seen as a reverse statement of the Young inequality for sparse convolutions. Our result hinges on a compression argument in additive set theory.

1. Introduction

Additive problems have increasingly become a focus in combinatorics, number theory, group theory and Fourier analysis as pointed out, e.g., in the textbook of Tao and Vu [14]. The key hereby is an understanding of the additive structure of finite subsets of an Abelian group G. The main result in the herein presented work is the application of a recent compression result in additive set theory by Grynkiewicz [6, Theorem 20.10] to sparse convolutions on discrete Abelian groups which are *torsion-free*, i.e., for any $N \in \mathbb{N}$ and $g \in G$ it holds $Ng = g + \cdots + g = 0$ if and only if $g = 0$. Compressing the convolution of sparse sequences, i.e., sequences with finite support sets, reduces to a compression of the sumset of their supports, since $\text{supp}(x * y) \subseteq \text{supp}(x) + \text{supp}(y)$. Our compression result allows to obtain a reverse statement of Young's

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inequality [18,17] for the convolution of all sparse $(x, y) \in \ell^2(G) \times \ell^1(G)$ in the form of

$$\|x * y\|_2 \geq \alpha \|x\|_2 \|y\|_1, \tag{1}$$

where $\alpha > 0$ exists and can be given in terms of the cardinalities of the supports of x and y.

For convolutions on generic locally compact Abelian groups, or LCA groups for short, such a reverse statement does not hold in general. To see this, take for any $N \in \mathbb{N}$ the finite group $\mathbb{Z}_N$. Full cancellation of the convolution occurs, i.e., $x * y = 0$, if we set $x = \delta_0 - \delta_1$ and $y = \sum_{i=0}^{N-1} \delta_i$, where for each $i \in \mathbb{Z}_N$ the sequence δ_i is given element-wise by $\delta_i(j) = 1$ if $i = j$ and 0 else. For the torsion-free group $\mathbb{Z}$ it is seen easily that a finite support length or sparsity prior excludes such cancellations. Note, that a sparsity prior on x and y is not of benefit for torsion groups. E.g., for G of even order N, we have for the 2-sparse sequences $x = \delta_0 + \delta_{N/2}$ and $y = \delta_0 - \delta_{N/2}$ full cancellation.

On arbitrary LCA groups the sharp upper bounds and maximizers for Young's inequalities are known [3,5], but lower bounds for the reverse case have only been shown for positive functions so far [10,5,2]. In this work we will give lower bounds $\alpha = \alpha(s, f)$ for arbitrary functions (sequences) depending only on their support length s respectively f. This is a universal result and translates to a weak stability for sparse convolutions, i.e., every f-sparse sequence y induces a convolution map $\cdot * y$ which is invertible over all s-sparse sequences x. Furthermore, this allows identifiability of sufficiently sparse auto-convolutions, since it holds

$$x * x - y * y = (x + y) * (x - y)$$

as shown by the first and second authors named in [16].

The article is structured as follows: Theorem 1 shows that sparse convolutions over torsion-free discrete Abelian groups can be represented by convolutions over $\mathbb{Z}$ with support contained in the first n integers. The integer n only depends on sparsity levels s and f of the convolution factors and not on the location or additive structure of the supports of the functions.

This compressed representation guarantees the existence of a lower norm bound $\alpha(s, f) > 0$ in (1). Theorem 2 can be seen as a smallest restricted eigenvalue property of all (s, f)-sparse convolutions. In Theorem 3 we give an analytical lower bound $\alpha(s, f, n)$, which scales exponentially in s and f and polynomial in n. Finally we will show that indeed there exist sparse sequences, given by uniform samples of a Gaussian and a modulated Gaussian, that produce numerical evidence for an exponential decay of the lower bound in the sparsity as derived in Theorem 3.

2. Notation

We will consider in this contribution only topological groups $G = (G, +, \mathcal{O})$ which have a locally compact topology $\mathcal{O}$ and are Abelian with group operation written additively, that is LCA groups. The up to normalization unique Haar measure of the LCA group G is denoted by μ. For $1 \leq p < \infty$ we denote by $L^p(G, \mu)$ the Banach space of all complex valued functions $x : G \to \mathbb{C}$ such that $\|x\|_p^p := \int_G |x(g)|^p d\mu(g) < \infty$. The convolution of $x, y \in L^1(G, \mu)$ is given by the formula

$$(x * y)(g) = \int_G x(g) y(h - g) d\mu(g) \tag{2}$$

for μ-almost every $g \in G$, see, e.g., [13, Section 1.1]. If $1 \leq p, q, r \leq \infty$ and $1/p + 1/q = 1 + 1/r$, then (2) can be defined for $x \in L^p(G, \mu)$ and $y \in L^q(G, \mu)$ and the *Young's inequality*

$$\|x * y\|_r \leq \|x\|_p \|y\|_q \tag{3}$$