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# Simplified vanishing moment criteria for wavelets over general dilation groups, with applications to abelian and shearlet dilation groups

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We consider the coorbit theory associated to a square-integrable, irreducible quasi-regular representation of a semidirect product group $G = \mathbb{R}^d \rtimes H$. The existence of coorbit spaces for this very general setting has been recently established, together with concrete vanishing moment criteria for analyzing vectors and atoms that can be used in the coorbit scheme. These criteria depend on fairly technical assumptions on the dual action of the dilation group, and it is one of the chief purposes of this paper to considerably simplify these assumptions.

We then proceed to verify the assumptions for large classes of dilation groups, in particular for all abelian dilation groups in arbitrary dimensions, as well as a class called *generalized shearlet dilation groups*, containing and extending all known examples of shearlet dilation groups employed in dimensions two and higher. We explain how these groups can be systematically constructed from certain commutative associative algebras of the same dimension, and give a full list, up to conjugacy, of shearing groups in dimensions three and four. In the latter case, three previously unknown groups are found.

As a result, the existence of Banach frames consisting of compactly supported wavelets, with simultaneous convergence in a whole range of coorbit spaces, is established for all groups involved.



## 1. Introduction: wavelet coorbit theory in higher dimensions

Coorbit theory can be understood as a group-theoretic formalism for the description of approximation-theoretic properties of building blocks arising from a unitary group action. It was initially developed with the aim to provide a unified view of large classes of function spaces, including the family of Besov spaces on one hand, with the underlying group given by the $ax + b$-group, and the modulation spaces,

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associated to a unitary action of the Heisenberg group, on the other. Later it was seen to apply to other settings and groups as well, for example the shearlet groups in dimensions two and higher [8,10,7]. This paper continues work begun in [25,26], which provided explicit criteria for wavelets associated to general dilation groups, ensuring their suitability as analyzing vectors and/or atoms in the coorbit scheme. The results in these sources depend on a number of fairly technical conditions on the dilation groups, defined as *(strong) temperate embeddedness* of the associated open dual orbit. It is the chief purpose of this paper to provide simple criteria for these conditions to hold, and to verify these for large classes of dilation groups. In particular, our results cover all shearlet dilation groups that have been considered so far, thus extending the known results for these groups in a unified manner.

### *1.1. Notation and preliminaries*

Before we describe the aims of this paper in more detail, let us quickly fix some notation that will be used throughout. We let $\mathbb{R}^+$ denote the set of strictly positive real numbers. $|\cdot| : \mathbb{R}^d \to \mathbb{R}$ denotes the Euclidean norm. Given a matrix $h \in \mathbb{R}^{d\times d}$, the operator norm of the induced linear map $(\mathbb{R}^d, |\cdot|) \to (\mathbb{R}^d, |\cdot|)$ is denoted by $\|h\|$. By a slight abuse of notation we use $|\alpha| = \sum_{i=1}^d \alpha_i$ for multiindices $\alpha \in \mathbb{N}_0^d$.

Given $f \in \mathrm{L}^1(\mathbb{R}^d)$, its Fourier transform is defined as

$$\mathcal{F}(f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x,\xi\rangle} dx ,$$

with $\langle \cdot,\cdot \rangle$ denoting the Euclidean scalar product on $\mathbb{R}^d$. We let $\mathcal{S}(\mathbb{R}^d)$ denote Schwartz space. Related to this space is the family of Schwartz norms, defined for $r, m > 0$ by

$$|f|_{r,m} = \sup_{x\in\mathbb{R}^d, |\alpha|\le r} (1 + |x|)^m |\partial^\alpha f(x)|$$

for any function $f : \mathbb{R}^d \to \mathbb{C}$ with suitably many partial derivatives.

$\mathcal{S}'(\mathbb{R}^d)$ denotes the dual of $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions. We denote the extension of the Fourier transform to $\mathcal{S}'(\mathbb{R}^d)$ by the same symbols as for $\mathrm{L}^1$-functions. For any subspace $X \subset \mathcal{S}'(\mathbb{R}^d)$, we let $\mathcal{F}^{-1}X$ denote its inverse image under the Fourier transform.

In order to avoid cluttered notation, we will occasionally use the expression $X \preceq Y$ between terms $X, Y$ involving one or more functions or vectors in $\mathbb{R}^d$, to indicate the existence of a constant $C > 0$, independent of the functions and vectors occurring in $X$ and $Y$, such that $X \le CY$.

Our conventions regarding locally compact groups, Haar measure etc. are the same as in Folland's book [19]. We fix a closed matrix group $H < \mathrm{GL}(d,\mathbb{R})$, the so-called **dilation group**, and let $G = \mathbb{R}^d \rtimes H$. This is the group of affine mappings generated by $H$ and all translations. Elements of $G$ are denoted by pairs $(x, h) \in \mathbb{R}^d \times H$, and the product of two group elements is given by $(x, h)(y, g) = (x + hy, hg)$. The left Haar measure of $G$ is given by $d(x, h) = |\det(h)|^{-1} dx dh$, and the modular function of $G$ is given by $\Delta_G(x, h) = \Delta_H(h)|\det(h)|^{-1}$.

$G$ acts unitarily on $\mathrm{L}^2(\mathbb{R}^d)$ by the **quasi-regular representation** defined by

$$[\pi(x, h)f](y) = |\det(h)|^{-1/2} f\left(h^{-1}(y - x)\right) . \quad (1)$$

### *1.2. Continuous wavelet transforms in higher dimensions*

Let us now shortly describe the ingredients going into the construction of higher-dimensional continuous wavelet transforms. For more background on the representation-theoretic aspects we refer to the book [23], and to the previous papers [25,26] for additional information concerning coorbit theory for this setting.

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