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A multifractal formalism for non-concave and non-increasing spectra: The leaders profile method

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A B S T R A C T

We present an implementation of a multifractal formalism based on the types of histogram of wavelet leaders. This method yields non-concave spectra and is not limited to their increasing part. We show both from the theoretical and from the applied points of view that this approach is more efficient than the wavelet-based multifractal formalisms previously introduced.

1. Introduction

Multifractal analysis initially appeared in the context of fully developed turbulence [1]; the aim was to study the Hölder regularity of a signal. Let us be more precise about this notion. A locally bounded function $f : \mathbf{R}^n \to \mathbf{R}$ belongs to the Hölder space $\Lambda^\alpha(x_0)$ (with $x_0 \in \mathbf{R}^n$ and $\alpha \geq 0$) if there exist a constant $C > 0$ and a polynomial P_{x_0} of degree less than α such that

$$|f(x) - P_{x_0}(x)| < C|x - x_0|^\alpha \tag{1}$$

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in a neighborhood of x_0. The polynomial is clearly unique and one has $\Lambda^{\alpha+\varepsilon}(x_0) \subset \Lambda^{\alpha}(x_0)$ for any $\varepsilon \geq 0$. The Hölder exponent of f at x_0 is defined by

$$h_f(x_0) = \sup\{\alpha \geq 0 : f \in \Lambda^{\alpha}(x_0)\}.$$

As usual, a function f belongs to $\Lambda^{\alpha}(\mathbf{R}^n)$ if f belongs to $\Lambda^{\alpha}(x_0)$ for any x_0, the constant C in (1) being uniform.

If the signal f is highly irregular, determining $h_f(x_0)$ for each x_0 is meaningless, since the function h_f can be itself very irregular. One tries instead to estimate the "size" of the iso-Hölder sets E_h, that is the sets of points sharing the same Hölder exponent h:

$$E_h = \{x \in \mathbf{R}^n : h_f(x) = h\}.$$

Such sets can be fractal sets, therefore by "size" one usually means Hausdorff dimension $\dim_{\mathcal{H}}$ (see e.g. [2]). The Hölder spectrum of f defined as

$$d_f : [0, +\infty] \to \{-\infty\} \cup [0, n] \quad h \mapsto \dim_{\mathcal{H}} E_h,$$

gives global information about the pointwise regularity of f.

Computing the Hölder spectrum by directly using the definition given above truly is an unattainable goal in most of the practical cases, but there exist heuristic methods to estimate d_f that give satisfactory results in many situations. Such a procedure is called a multifractal formalism; if it leads to the exact spectrum of the function f, one says that the multifractal formalism is satisfied for f. A method was first proposed by Parisi and Frisch [1]; later, Arneodo et al. proposed a similar method based on the continuous wavelet transform [3]. In both approaches, one tries to compute the function

$$\eta(q) = \sup\{s : f \in B^{s/q}_{q,\infty}\} \tag{2}$$

relying on the Besov spaces $B^s_{q,p}$ using the box-counting technique or wavelets [4], since heuristic arguments underlie the following equality:

$$d_f(h) = \inf_q \{hq - \eta(q)\} + n.$$

From a mathematical point of view, these methods only lead to an upper bound [4]. More precisely, if $f : \mathbf{R}^n \to \mathbf{R}$ is a locally bounded function belonging to $\Lambda^{\varepsilon}(\mathbf{R}^n)$ for some $\varepsilon > 0$, then

$$d_f(h) \leq \inf_{q > q_0} \{hq - \eta(q)\} + n,$$

where q_0 satisfies $\eta(q_0) = n$.

Since the Besov spaces $B^s_{q,p}$ are not defined for negative values of q, the decreasing part of the spectrum cannot be obtained using the method described above. To take care of this problem, Arneodo et al. proposed the wavelet transform modulus maxima (WTMM) method [5], using the notion of line of maxima in the wavelet transform. This technique proved helpful in many practical problems, but its theoretical contribution was limited; in particular, there is no underlying functional space. This is why Jaffard replaced the continuous wavelet transform with the discrete one and introduced the wavelet leaders method (WLM) [6], based on the oscillation spaces O^s_q [7]. These spaces are well defined for negative values of q and $s > n/q$ implies $O^s_q = B^s_{q,\infty}$. The idea consists in replacing Besov spaces in (2) with oscillation spaces to get

$$\eta(q) = \sup\{s : f \in O^{s/q}_q\}.$$

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