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# Fully discrete needlet approximation on the sphere ☆

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ABSTRACT

Spherical needlets are highly localized radial polynomials on the sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, $d \geq 2$, with centers at the nodes of a suitable cubature rule. The original semidiscrete spherical needlet approximation of Narcowich, Petrushev and Ward is not computable, in that the needlet coefficients depend on inner product integrals. In this work we approximate these integrals by a second quadrature rule with an appropriate degree of precision, to construct a fully discrete needlet approximation. We prove that the resulting approximation is equivalent to filtered hyperinterpolation, that is to a filtered Fourier–Laplace series partial sum with inner products replaced by appropriate cubature sums. It follows that the $\mathbb{L}_p$-error of discrete needlet approximation of order $J$ for $1 \leq p \leq \infty$ and $s > d/p$ has for a function $f$ in the Sobolev space $\mathbb{W}_p^s(\mathbb{S}^d)$ the optimal rate of convergence in the sense of optimal recovery, namely $\mathcal{O}\left(2^{-Js}\right)$. Moreover, this is achieved with a filter function that is of smoothness class $C^{\left\lfloor \frac{d+3}{2} \right\rfloor}$, in contrast to the usually assumed $C^\infty$. A numerical experiment for a class of functions in known Sobolev smoothness classes gives $\mathbb{L}_2$ errors for the fully discrete needlet approximation that are almost identical to those for the original semidiscrete needlet approximation. Another experiment uses needlets over the whole sphere for the lower levels together with high-level needlets with centers restricted to a local region. The resulting errors are reduced in the local region away from the boundary, indicating that local refinement in special regions is a promising strategy.



## 1. Introduction

Spherical wavelets [13] have wide applications in areas such as signal processing [20], geography [12,35,36] and cosmology [21,33,41]. The classical continuous wavelets represent a complicated function by projecting it onto different levels of a decomposition of the $\mathbb{L}_2$ function space on the sphere. A projection, often called

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"a detail of the function", becomes small rapidly as the level increases. This multilevel decomposition proves very useful in solving many problems.

Narcowich et al. in recent work [29,30] showed that the details of the spherical wavelets may be further broken up into still finer details, which are highly localized in space. This new decomposition of a spherical function is said to be a needlet decomposition.

Needlet approximation in its original form is however not suitable for direct implementation as its needlet coefficients are integrals. In this paper, we introduce a *discrete* spherical needlet approximation scheme by using spherical quadrature rules to approximate the inner product integrals and establish its approximation error for functions in Sobolev spaces on the sphere. Numerical experiments are carried out for this fully discrete version of the spherical needlet approximation.

Before we describe spherical needlets and the discrete spherical needlet approximation we need some definitions. For $d \ge 2$, let $\mathbb{R}^{d+1}$ be the real $(d+1)$-dimensional Euclidean space with inner product $\mathbf{x}\cdot\mathbf{y}$ for $\mathbf{x},\mathbf{y} \in \mathbb{R}^{d+1}$ and Euclidean norm $|\mathbf{x}| := \sqrt{\mathbf{x}\cdot\mathbf{x}}$. Let $\mathbb{S}^d := \{\mathbf{x} \in \mathbb{R}^{d+1} : |\mathbf{x}| = 1\}$ denote the unit sphere of $\mathbb{R}^{d+1}$. The sphere $\mathbb{S}^d$ forms a compact metric space, with the metric being the geodesic distance $\mathrm{dist}(\mathbf{x},\mathbf{y}) := \arccos(\mathbf{x}\cdot\mathbf{y})$ for $\mathbf{x},\mathbf{y} \in \mathbb{S}^d$.

For $1 \le p \le \infty$ let $\mathbb{L}_p(\mathbb{S}^d) = \mathbb{L}_p(\mathbb{S}^d, \sigma_d)$, endowed with the norm $\|\cdot\|_{\mathbb{L}_p(\mathbb{S}^d)}$, be the real $\mathbb{L}_p$-function space on $\mathbb{S}^d$ with respect to the normalized Riemann surface measure $\sigma_d$ on $\mathbb{S}^d$. For $p = 2$, $\mathbb{L}_2(\mathbb{S}^d)$ forms a Hilbert space with inner product $(f,g)_{\mathbb{L}_2(\mathbb{S}^d)} := \int_{\mathbb{S}^d} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{x})$ for $f,g \in \mathbb{L}_2(\mathbb{S}^d)$.

A *spherical harmonic* of degree $\ell$ on $\mathbb{S}^d$ is the restriction to $\mathbb{S}^d$ of a homogeneous and harmonic polynomial of total degree $\ell$ defined on $\mathbb{R}^{d+1}$. Let $\mathcal{H}_\ell^d$ denote the set of all spherical harmonics of exact degree $\ell$ on $\mathbb{S}^d$. The dimension of the linear space $\mathcal{H}_\ell^d$ is

$$Z(d,\ell) := (2\ell + d - 1)\frac{\Gamma(\ell + d - 1)}{\Gamma(d)\Gamma(\ell + 1)} \asymp (\ell + 1)^{d-1}, \tag{1}$$

where $\Gamma(\cdot)$ denotes the gamma function and $a_\ell \asymp b_\ell$ means $c\, b_\ell \le a_\ell \le c'\, b_\ell$ for some positive constants $c$, $c'$, and the asymptotic estimate uses [9, Eq. 5.11.12]. The linear span of $\mathcal{H}_\ell^d$, $\ell = 0, 1, \ldots, \nu$ forms the space $\mathbb{P}_\nu(\mathbb{S}^d)$ of spherical polynomials of degree up to $\nu$. Let $P_\ell^{(\alpha,\beta)}$ be the Jacobi polynomial of degree $\ell$ for $\alpha, \beta > -1$. We denote the normalized Legendre or Gegenbauer polynomial by

$$P_\ell^{(d+1)}(t) := P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(t)/P_\ell^{(\frac{d-2}{2},\frac{d-2}{2})}(1). \tag{2}$$

Given $N \ge 1$, for $k = 1, \ldots, N$, let $\mathbf{x}_k$ be $N$ nodes on $\mathbb{S}^d$ and let $w_k > 0$ be corresponding weights. The set $\{(w_k, \mathbf{x}_k) : k = 1, \ldots, N\}$ is a *positive quadrature (numerical integration) rule* exact for polynomials of degree up to $\nu$ for some $\nu \ge 0$ if

$$\int_{\mathbb{S}^d} p(\mathbf{x})\,\mathrm{d}\sigma_d(\mathbf{x}) = \sum_{k=1}^{N} w_k\, p(\mathbf{x}_k), \quad \text{for all } p \in \mathbb{P}_\nu(\mathbb{S}^d).$$

*Spherical needlets* [29,30] are a type of localized polynomial on the sphere associated with a *quadrature rule* and a *filter*. Let $\mathbb{R}_+ := [0, +\infty)$.

**Definition 1.1.** A continuous compactly supported function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a filter. We will only consider a filter with support a subset of $[0, 2]$.

A filtered kernel on $\mathbb{S}^d$ with filter $g$ is, for $T \in \mathbb{R}_+$,

$$v_{T,g}(\mathbf{x}\cdot\mathbf{y}) := v_{T,g}^d(\mathbf{x}\cdot\mathbf{y}) := \begin{cases} 1, & 0 \le T < 1, \\ \displaystyle\sum_{\ell=0}^{\infty} g\left(\frac{\ell}{T}\right) Z(d,\ell)\, P_\ell^{(d+1)}(\mathbf{x}\cdot\mathbf{y}), & T \ge 1. \end{cases} \tag{3}$$