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Letter to the Editor

Explicit universal sampling sets in finite vector spaces

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A B S T R A C T

In this paper we construct explicit sampling sets and present reconstruction algorithms for Fourier signals on finite vector spaces G, with $|G| = p^r$ for a suitable prime p. The two sampling sets have sizes of order $O(pt^2r^2)$ and $O(pt^2r^3 \log(p))$ respectively, where t is the number of large coefficients in the Fourier transform. The algorithms approximate the function up to a small constant of the best possible approximation with t non-zero Fourier coefficients. The fastest of the algorithms has complexity $O(p^2t^2r^3 \log(p))$.

1. Introduction

The problem of compressive sensing originated in the context of Fourier series [3]. The aim is to reconstruct a linear combination of a small number of complex exponentials from as few samples as possible, when only the number of the exponentials entering the linear combination is known. The additional challenge was to come up with practical and efficient methods for the reconstruction (which by its combinatorial nature is NP-hard, unless extra information is available).

Later on, the compressed sensing problem evolved to include a more general setup. The overall problems and main challenges, however, remained the same; and they concerned mostly the construction of sampling schemes that would allow (and guarantee) efficient reconstruction from as few measurements as possible, and the design of efficient reconstruction algorithms. For the latter, ℓ^1-minimization turned out to be a popular choice, and the chief technical condition to guarantee success for the reconstruction method was the *restricted isometry condition* (see [4] for the first introduction of these ideas, and [6] for an in-depth study). However, there still remained the problem of constructing measurement matrices (or, in the Fourier case, sampling sets) for which the RIP was actually provably fulfilled. An important (somewhat partial) answer to this problem was provided by random methods; e.g., in the case of random sampling of Fourier matrices the RIP assumption turns out to be true under rather weak assumptions on the number of samples [13], at least with high probability.

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However, the case of *deterministic sampling sets* with *provably* guaranteed reconstruction poses altogether different challenges. Firstly, the verification of properties like RIP is a very complex problem by itself [14], hence special care must be taken to allow such estimates. A first successful example for a deterministic construction with the RIP property was presented by DeVore [5]. For the Fourier setting, in [1,8] sampling sets and inversion algorithms were constructed for cyclic groups; an alternative construction of deterministic sampling sets guaranteeing RIP for the cyclic case was developed in [7]. All these constructions have a common restriction, commonly known as *quadratic bottleneck*: In order to reconstruct linear combinations of t basis vectors, they need $O(t^2)$ samples. A more recent paper by Bourgain and collaborators [2] managed to improve this to $O(t^{2-\epsilon})$, for a very small $\epsilon > 0$, using rather involved arguments from additive combinatorics.

This paper considers efficiently sampling Fourier-sparse vectors on finite abelian groups. It can be seen as complementary to [1,7,8], with the main difference being that this paper focuses on *finite vector spaces* rather than cyclic groups. We develop a general, simple scheme for the design of sampling sets, together with algorithms that allow reconstruction.

Hence the new methods provide an alternative means of explicitly designing *universal sampling sets* in a specific family of finite abelian groups G, i.e., sampling sets $\Omega \subset G$ that allow the reconstruction of any given linear combination of t characters of G from its restriction on Ω, together with *explicit* inversion algorithms, both for the noisy and noise free cases. The groups G we consider are finite vector spaces, and the sampling sets will be written as unions of suitable affine subspaces. The sampling sets actually fulfill the RIP property, which allows one to use the standard methods such as ℓ^1-minimization. However, the special structure of the sampling set makes the inversion algorithm particularly amenable to the use of a more structured (and potentially faster) reconstruction algorithm, using FFT methods. It should be stressed, though, that our construction is *not* able to beat the quadratic bottleneck.

2. Notation

Let p be a prime and r a positive integer. When considering their additive groups structure we have that $(\mathbb{Z}/p\mathbb{Z})^r \cong \mathbb{F}_p^r$. The vector space structure of $\mathbb{F}_p^r$ will enable us to construct the sampling sets needed in the algorithms described in this paper. We will write $\mathbb{F}_p^r$ also when considering only its additive group structure. Also in order to avoid complicated notations we will identify, where needed, elements of $\mathbb{Z}$ with their images in $\mathbb{F}_p$, so that for example 1 could be viewed as either an element of $\mathbb{Z}$ or of $\mathbb{F}_p$.

We will write $H \leq G$ for a subgroup H of G. Subsets of $\mathbb{F}_p^r$ are subgroups if and only if they are vector spaces (since $\mathbb{F}_p$ is a field of prime order). So when looking at $\mathbb{F}_p^r$ as a vector space we will write $H \leq \mathbb{F}_p^r$ for a subspace H. For any subgroup $H \leq G$ we will also write $\mathrm{Rep}(G/H)$ for a set of representatives of cosets of H in G.

In Section 4 we will also be working with both $\mathbb{F}_p$ and $\mathbb{F}_q$, where q is a power of p. Since vector spaces over $\mathbb{F}_q$ can be viewed also as vector spaces over $\mathbb{F}_p$, we will write $\dim_{\mathbb{F}_p}(V)$ and $\dim_{\mathbb{F}_q}(V)$ for the dimension of V as a vector space over $\mathbb{F}_p$ or $\mathbb{F}_q$ respectively. Similarly we will write $\mathrm{span}_{\mathbb{F}_p}(A)$ and $\mathrm{span}_{\mathbb{F}_q}(A)$ for the span of A as vector space over $\mathbb{F}_p$ or $\mathbb{F}_q$ respectively.

Let $\widehat{\mathbb{F}_p^r}$ consist of all group homomorphisms $\mathbb{F}_p^r \to \mathbb{C}$. We have that

$$\widehat{\mathbb{F}_p^r} = \{\chi_{(y_1,\ldots,y_r)} : y_i \in \mathbb{F}_p\} = \{\chi_y : y \in \mathbb{F}_p^r\},$$

where, if $\omega_p = e^{2\pi i/p}$, we define $\chi_{(y_1,\ldots,y_r)}(x_1,\ldots,x_r) := \omega_p^{x_1y_1+\ldots+x_ry_r}$ for $(x_1,\ldots,x_r) \in \mathbb{F}_p^r$. This is well defined since $\omega_p^p = 1$. Also it is easy to check that $\chi_y\chi_z = \chi_{y+z}$ for $y, z \in \mathbb{F}_p^r$.

For any function $f : (\mathbb{Z}/p\mathbb{Z})^r \to \mathbb{C}$ let $\widehat{f} : \widehat{\mathbb{F}_p^r} \to \mathbb{C}$ be its Fourier transform, so that

$$f = \sum_{\chi_y \in \widehat{\mathbb{F}_p^r}} \widehat{f}(\chi_y)\chi_y.$$

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