A sampling theory for non-decaying signals

Ha Q. Nguyen, Michael Unser

PII: S1063-5203(15)00147-5
DOI: http://dx.doi.org/10.1016/j.acha.2015.10.006
Reference: YACHA 1092

To appear in: *Applied and Computational Harmonic Analysis*

Received date: 22 April 2015
Accepted date: 6 October 2015

Please cite this article in press as: H.Q. Nguyen, M. Unser, A sampling theory for non-decaying signals, *Appl. Comput. Harmon. Anal.* (2015), http://dx.doi.org/10.1016/j.acha.2015.10.006

A Sampling Theory for Non-Decaying Signals☆

Ha Q. Nguyen*, Michael Unser

Biomedical Imaging Group, École Polytechnique Fédérale de Lausanne (EPFL), Station 17, CH-1015, Lausanne, Switzerland

Abstract

The classical assumption in sampling and spline theories is that the input signal is square-integrable, which prevents us from applying such techniques to signals that do not decay or even grow at infinity. In this paper, we develop a sampling theory for multidimensional non-decaying signals living in weighted L_p spaces. The sampling and reconstruction of an analog signal can be done by a projection onto a shift-invariant subspace generated by an interpolating kernel. We show that, if this kernel and its biorthogonal counterpart are elements of appropriate hybrid-norm spaces, then both the sampling and the reconstruction are stable. This is an extension of earlier results by Aldroubi and Gröchenig. The extension is required because it allows us to develop the theory for the ideal sampling of non-decaying signals in weighted Sobolev spaces. When the d-dimensional signal and its $d/p+\varepsilon$ derivatives, for arbitrarily small $\varepsilon > 0$, grow no faster than a polynomial in the L_p sense, the sampling operator can be showed to be bounded even without a sampling kernel. As a consequence, the signal can also be interpolated from its samples with a nicely behaved interpolating kernel.

Keywords: sampling theory, shift-invariant spaces, spline interpolation, weighted L_p spaces, weighted Sobolev spaces, hybrid-norm spaces, Wiener amalgams

1. Introduction

The sampling theory has a rich history [1, 2] starting with the classical Whittaker-Shannon-Kotel'nikov sampling theorem [3]. The crucial fact behind the sampling theorem is that bandlimited signals live in the *shift-invariant (spline-like) space* generated by the *sinc* function. Sampling theory has been extended for the general shift-invariant spaces generated by splines or wavelets, which exhibit better localization properties than the ideal sinc kernel [4, 5, 6]. Although a large body of work has been dedicated to sampling in shift-invariant spaces, [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] just to name a few, a general theory for sampling *non-decaying* signals seems to be still missing. There were some attempts to generalize the sampling theorem for bandlimited signals of polynomial growth [21, 22, 23, 24], but the bandlimitedness requirement is too restrictive. Aldroubi and Gröchenig developed in [10] a theory for sampling signals in weighted L_p spaces with *moderate* weights. Although the authors did not emphasize this property, their theoretical framework can also handle the sampling of growing signals with a well-behaved sampling kernel as moderate weights might be decaying. However, in the absence of a prefilter, the *ideal* sampling of weighted L_p signals cannot be done stably. To the best of our knowledge, none of the existing theories apply to the ideal sampling of general functions that are non-decreasing, such as realizations of Brownian and Lévy processes, which happen to be intimately linked to splines [25, 26, 27]. Intuitively, however, there seems to be no fundamental reason to prevent one from sampling a continuous signal even if it is not decaying.

In the first half of this paper, we develop a theory for the *regular* sampling of *multidimensional* non-decaying signals that are modeled as elements of weighted L_p spaces in which the growth of the signals is

☆This research was funded by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant Agreement n° 267439 and the Swiss National Science Foundation under Grant 200020-162343.

*Corresponding author

Email addresses: ha.nguyen@epfl.ch (Ha Q. Nguyen), michael.unser@epfl.ch (Michael Unser)