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Gabor frames of Gaussian beams for the Schrödinger equation

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A B S T R A C T

The present paper is devoted to the semiclassical analysis of linear Schrödinger equations from a Gabor frame perspective. We consider (time-dependent) smooth Hamiltonians with at most quadratic growth. Then we construct higher order parametrices for the corresponding Schrödinger equations by means of $\hbar$-Gabor frames, as recently defined by M. de Gosson, and we provide precise L^2-estimates of their accuracy, in terms of the Planck constant $\hbar$. Nonlinear parametrices, in the spirit of the nonlinear approximation, are also presented. Numerical experiments are exhibited to compare our results with the early literature.

1. Introduction

The goal of this paper is to construct asymptotic solutions for Schrödinger equations

$$\begin{cases} i\hbar\partial_t u = \widehat{H(t)}u \\ u(0) = u_0, \end{cases} \tag{1}$$

by means of Gabor frames in the semiclassical regime ($\hbar \to 0^+$). Here $t \in [0, T]$, the initial condition $u_0 \in L^2(\mathbb{R}^d)$ and the quantum Hamiltonian $\widehat{H(t)}$ is supposed to be the $\hbar$-Weyl quantization of the classical observable $H(t, X)$, with $X = (x, \xi) \in \mathbb{R}^{2d}$.

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1.1. Literature overview

There are many results about asymptotic solutions for partial differential equations (PDE's), especially when the initial value is a wave packet, i.e. it is well localized in the physical space and it oscillates with an approximately constant frequency. In particular, if the initial profile is Gaussian (a coherent state), the solution will be highly concentrated along the classical trajectory, according to the correspondence principle. Such a semiclassical analysis for Schrödinger-type equations were widely studied in several papers, see e.g. [6,17,19,30–32,41] and the textbooks [7,27–29,46].

The natural idea of this work is to decompose the initial value u_0 in (1) by means of a $\hbar$-Gabor frame whose atoms are Gaussian coherent states, construct asymptotic solutions for each of them, a so-called *Gaussian beam*, and finally by superposition obtain the asymptotic solution to (1). The main issues are the following:

- Construction of a parametrix via Gabor frames.
- Estimates in L^2 for the parametrix and the error term.
- Numerical results.

Despite the simplicity of the idea, we do not know a fully rigorous treatment of this matter. There are various attempts (see e.g. [2] and references therein) where however several arguments are carried out only at a heuristic level and with numerical experiments. The present paper is devoted to a rigorous study of these issues for a class of smooth Hamiltonians with at most quadratic growth and, unlike the previous work, we address from the beginning a finer analysis, that is higher order approximations: the approximate solution is searched as a (finite) sum of powers of $\hbar$, and the order of approximation can be arbitrarily large.

1.2. Notation and ($\hbar$-)Gabor frames

To be explicit, let us fix some notation.

The $\hbar$-Weyl quantization of a function H on the phase space $\mathbb{R}^{2d}$ is formally defined by

$$\widehat{H}u(x) = Op^w_\hbar[H]u(x) = (2\pi\hbar)^{-d} \int_{\mathbb{R}^{2d}} e^{i\hbar^{-1}(x-y)p} H\Big(\frac{x+y}{2}, p\Big) u(y)\, dydp \tag{2}$$

for every u in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. The function H is called the $\hbar$-Weyl symbol of $\widehat{H}$. For $z_0 = (x_0, p_0) \in \mathbb{R}^{2d}$, we define by $\widehat{\mathcal{T}}^\hbar(z_0)$ the Weyl operator

$$\widehat{\mathcal{T}}^\hbar(z_0)u(x) = Op^w_\hbar[e^{i\hbar^{-1}(p_0x - x_0p)}]u(x) = e^{i\hbar^{-1}(p_0x - x_0p_0/2)}u(x - x_0). \tag{3}$$

Such operator meets the definition of the so-called $\hbar$-Gabor frames, introduced in [19] as generalizations of Gabor frames. For a given lattice Λ in $\mathbb{R}^{2d}$ and a non-zero square integrable function φ (called window) on $\mathbb{R}^d$ the system

$$\mathcal{G}^\hbar(\varphi, \Lambda) = \{\widehat{\mathcal{T}}^\hbar(z)\varphi \,:\, z \in \Lambda\}$$

is called a $\hbar$-Gabor frame if it is a frame for $L^2(\mathbb{R}^d)$, that is there exist constants $0 < a \le b$ such that

$$a\|f\|_2^2 \le \sum_{z\in\Lambda} |\langle f, \widehat{\mathcal{T}}^\hbar(z)\varphi\rangle|^2 \le b\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d). \tag{4}$$

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