ARTICLE IN PRESS
YACHA:1113

Appl. Comput. Harmon. Anal. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Letter to the Editor

Frames of directional wavelets on n-dimensional spheres

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A R T I C L E   I N F O

Article history:
Received 14 May 2015
Received in revised form 11 January 2016
Accepted 15 January 2016
Available online xxxx
Communicated by Guy Battle

MSC:
42C40

Keywords:
Spherical wavelets
Approximate identities
Frames
n-Spheres

A B S T R A C T

The major goal of the paper is to prove that discrete frames of (directional) wavelets derived from an approximate identity exist. Additionally, a kind of energy conservation property is shown to hold in the case when a wavelet family is not its own reconstruction family. Although an additional constraint on the spectrum of the wavelet family must be satisfied, it is shown that all the wavelets so far defined in the literature possess this property.

1. Introduction

The present paper is a continuation of the series [11,13,14,12]. A wide class of zonal wavelets is constructed, which contains all the so far studied wavelet families (derived from approximate identities). For the case of non-zonal wavelets, the definition of the wavelet transform given in [11] is weakened, which is a price for having fully discrete directional wavelet frames. The frame constructions on n-dimensional spheres known to the author are all based on zonal wavelets [8,14], therefore, the present paper seems to be the first approach to discretize a directional wavelet transform.

The paper is organized as follows. In Section 2 we recapitulate basic facts about spherical functions, frames, and wavelets derived from approximate identities. We also present a wide class of functions that can serve as wavelet families. In Section 3 we show that the wavelet transform with respect to such wavelet families can be discretized with respect to the scale parameter, and in Section 4 we prove the existence of fully discrete wavelet frames.

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http://dx.doi.org/10.1016/j.acha.2016.01.004
1063-5203/

Please cite this article in press as: I. Iglewska-Nowak, Frames of directional wavelets on n-dimensional spheres, Appl. Comput. Harmon. Anal. (2016), http://dx.doi.org/10.1016/j.acha.2016.01.004

2. Preliminaries

2.1. Functions on the sphere

By $\mathcal{S}^n$ we denote the n-dimensional unit sphere in $n+1$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the rotation-invariant measure $d\sigma_n$ normalized such that

$$\Sigma_n = \int_{\mathcal{S}^n} d\sigma_n = \frac{2\pi^{(n+1)/2}}{\Gamma\big((n+1)/2\big)}.$$

The surface element $d\sigma_n$ is explicitly given by

$$d\sigma_n = \sin^{n-1}\theta_1 \, \sin^{n-2}\theta_2 \ldots \sin\theta_{n-1} d\theta_1 \, d\theta_2 \ldots d\theta_{n-1} d\varphi,$$

where $(\theta_1, \theta_2, \ldots, \theta_{n-1}, \varphi) \in [0,\pi]^{n-1} \times [0, 2\pi)$ are spherical coordinates satisfying

$$\begin{aligned}
x_1 &= \cos\theta_1, \\
x_2 &= \sin\theta_1 \cos\theta_2, \\
x_3 &= \sin\theta_1 \sin\theta_2 \cos\theta_3, \\
&\ldots \\
x_{n-1} &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \cos\theta_{n-1}, \\
x_n &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \sin\theta_{n-1} \cos\varphi, \\
x_{n+1} &= \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{n-2} \sin\theta_{n-1} \sin\varphi.
\end{aligned}$$

$\langle x, y\rangle$ or $x \cdot y$ stand for the scalar product of vectors with origin in O and an endpoint on the sphere. As long as it does not lead to misunderstandings, we identify these vectors with points on the sphere. By their geodesic distance we mean $\angle(x, y) = \arccos(x, y)$.

A function is called zonal if its value depends only on $\theta = \theta_1 = \langle \hat{e}, x\rangle$, where $\hat{e}$ is the north pole of the sphere

$$\hat{e} = (1, 0, 0, \ldots, 0).$$

It is invariant with respect to the rotation about the axis through O and $\hat{e}$. We identify zonal functions with functions over the interval $[-1, 1]$, i.e., whenever it does not lead to mistakes, we write

$$f(x) = f(\cos\theta_1).$$

The subspace of p-integrable zonal functions is isomorphic to and will be identified with the space $\mathcal{L}^p_\lambda$, and the norms satisfy

$$\|f\|_{\mathcal{L}^p(\mathcal{S}^n)} = \|f\|_{\mathcal{L}^p_\lambda([-1,1])}$$

for

$$\|f\|_{\mathcal{L}^p(\mathcal{S}^n)} := \left[\frac{1}{\Sigma_n} \int_{\mathcal{S}^n} |f(x)|^p \, d\sigma_n(x)\right]^{1/p}$$

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