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Approximations in Sobolev spaces by prolate spheroidal wave functions ☆

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A B S T R A C T

Recently, there is a growing interest in the spectral approximation by the Prolate Spheroidal Wave Functions (PSWFs) $\psi_{n,c}$, $c > 0$. This is due to the promising new contributions of these functions in various classical as well as emerging applications from Signal Processing, Geophysics, Numerical Analysis, etc. The PSWFs form a basis with remarkable properties not only for the space of band-limited functions with bandwidth c, but also for the Sobolev space $H^s([-1,1])$. The quality of the spectral approximation and the choice of the parameter c when approximating a function in $H^s([-1,1])$ by its truncated PSWFs series expansion, are the main issues. By considering a function $f \in H^s([-1,1])$ as the restriction to $[-1,1]$ of an almost time-limited and band-limited function, we try to give satisfactory answers to these two issues. Also, we illustrate the different results of this work by some numerical examples.

1. Introduction

Let f be a function that belongs to some Sobolev space $H^s(I)$, $s > 0$, $I = [-1,1]$. The main issue of this work concerns the speed of convergence in $L^2(I)$ of its expansion in some PSWF basis.

Let us recall that, for a given value $c > 0$, called the bandwidth, PSWFs $(\psi_{n,c})_{n\geq 0}$ constitute an orthonormal basis of $L^2([-1,+1])$ of eigenfunctions of the two compact integral operators $\mathcal{F}_c$ and $\mathcal{Q}_c = \frac{c}{2\pi}\mathcal{F}_c^*\mathcal{F}_c$, defined on $L^2(I)$ by

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$$\mathcal{F}_c(f)(x) = \int_{-1}^{1} e^{i\,c\,x\,y} f(y)\,dy, \quad \mathcal{Q}_c(f)(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} f(y)\,dy. \tag{1}$$

PSWFs are also eigenfunctions of the Sturm–Liouville operator $\mathcal{L}_c$, defined by

$$\mathcal{L}_c(\psi) = -\frac{d}{d\,x}\left[(1-x^2)\frac{d\psi}{d\,x}\right] + c^2x^2\psi. \tag{2}$$

We call $\chi_n(c)$ the eigenvalues of $\mathcal{L}_c$, and $\lambda_n(c)$ the eigenvalues of $\mathcal{Q}_c$. The first ones are arranged in the increasing order, the second ones in the decreasing order $1 > \lambda_0(c) > \lambda_1(c) > \cdots > \lambda_n(c) > \cdots$. We finally call $\mu_n(c)$ the eigenvalues of $\mathcal{F}_c$. They are given by

$$\mu_n(c) = i^n\sqrt{\frac{2\pi}{c}\lambda_n(c)}.$$

By Plancherel identity, PSWFs are normalized so that

$$\int_{-1}^{1} |\psi_{n,c}(x)|^2\,dx = 1, \quad \int_{\mathbb{R}} |\psi_{n,c}(x)|^2\,dx = \frac{1}{\lambda_n(c)}, \quad n \geq 0. \tag{3}$$

We adopt the sign normalization of the PSWFs, given by

$$\psi_{n,c}(0) > 0 \ \text{ for even } \ n, \quad \psi'_{n,c}(0) > 0, \ \text{ for odd } \ n. \tag{4}$$

A breakthrough in the theory and the computation of the PSWFs goes back to the 1960s and is due to D. Slepian and his co-authors H. Landau and H. Pollak. For the classical and more recent developments in the area of the PSWFs, the reader is referred to the recent books on the subjects [13,16]. This paper is a companion paper of [3] and we refer to it for further notations and references.

This question of the quality of approximation has attracted a growing interest while, at the same time, were built PSWFs based numerical schemes for solving various problems from numerical analysis, see [4–6,12, 14,17,20]. In particular, in [4], the author has shown that a PSWF approximation based method outperforms in terms of spatial resolution and stability of time-step, the classical approximation methods based on Legendre or Tchebyshev polynomials. The authors of [6] were among the first to compare the quality of approximation by the PSWFs for different values of c. In particular, they have given an estimate of the decay of the PSWFs expansion coefficients of a function $f \in H^s(I)$, see also [4]. Recently, in [20], the author studied the speed of convergence of the expansion of such a function in a basis of PSWFs. We should mention that the methods used in the previous three references are heavily based on the use of the properties of the PSWFs as eigenfunctions of the differential operator $\mathcal{L}_c$, given by (2). They pose the problem of the best choice of the value of the band-width $c > 0$, for approximating well a given $f \in H^s(I)$, but their answer is mainly experimental. It has been numerically checked in [4,20] that the smaller the value of s, the larger the value of c should be.

Our study tries to give a satisfactory answer to this important problem of the choice of the parameter c. More precisely, we show that if $f \in H^s(I)$, for some positive real number $s > 0$, then for any integer $N \geq 1$, we have

$$\|f - S_N f\|_{L^2(I)} \leq K(1+c^2)^{-s/2}\|f\|_{H^s(I)} + K\sqrt{\lambda_N(c)}\|f\|_{L^2(I)}. \tag{5}$$

Here, $S_N f = \sum_{k=0}^{N} < f, \psi_{n,c} > \psi_{n,c}$ and K is a constant depending only on s. With this expression, one sees clearly how to distribute a fixed error between that part which is due to the smoothness of the function and