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# Dynamical sampling ☆

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Let $Y = \{f(i), Af(i), \ldots, A^{l_i}f(i) : i \in \Omega\}$, where $A$ is a bounded operator on $\ell^2(I)$. The problem under consideration is to find necessary and sufficient conditions on $A, \Omega, \{l_i : i \in \Omega\}$ in order to recover any $f \in \ell^2(I)$ from the measurements $Y$. This is the so-called dynamical sampling problem in which we seek to recover a function $f$ by combining coarse samples of $f$ and its futures states $A^l f$. We completely solve this problem in finite dimensional spaces, and for a large class of self adjoint operators in infinite dimensional spaces. In the latter case, although $Y$ can be complete, using the Müntz–Szász Theorem we show it can never be a basis. We can also show that, when $\Omega$ is finite, $Y$ is not a frame except for some very special cases. The existence of these special cases is derived from Carleson's Theorem for interpolating sequences in the Hardy space $H^2(\mathbb{D})$. Finally, using the recently proved Kadison–Singer/Feichtinger theorem we show that the set obtained by normalizing the vectors of $Y$ can never be a frame when $\Omega$ is finite.



## 1. Introduction

Dynamical sampling refers to the process that results from sampling an evolving signal $f$ at various times and asks the question: when do coarse samplings taken at varying times contain the same information as a finer sampling taken at the earliest time? In other words, under what conditions on an evolving system, can time samples be traded for spatial samples? Because dynamical sampling uses samples from varying

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time levels for a single reconstruction, it departs from classical sampling theory in which a signal $f$ does not evolve in time and is to be reconstructed from its samples at a single time $t = 0$, see [1,2,5,7,8,11,19,20,28,23,30,33,37,43,44], and references therein.

The general dynamical sampling problem can be stated as follows: Let $f$ be a function in a separable Hilbert space $\mathcal{H}$, e.g., $\mathbb{C}^d$ or $\ell^2(\mathbb{N})$, and assume that $f$ evolves through an evolution operator $A : \mathcal{H} \to \mathcal{H}$ so that the function at time $n$ has evolved to become $f^{(n)} = A^n f$. We identify $\mathcal{H}$ with $\ell^2(I)$ where $I = \{1, \dots, d\}$ in the finite dimensional case, $I = \mathbb{N}$ in the infinite dimensional case. We denote by $\{e_i\}_{i \in I}$ the standard basis of $\ell^2(I)$.

The time-space sample at time $t \in \mathbb{N}$ and location $p \in I$, is the value $A^t f(p)$. In this way we associate with each pair $(p, t) \in I \times \mathbb{N}$ a sample value.

The general dynamical sampling problem can then be described as: Under what conditions on the operator $A$, and a set $S \subset I \times \mathbb{N}$, can every $f$ in the Hilbert space $H$ be recovered in a stable way from the samples in $S$.

At time $t = n$, we sample $f$ at the locations $\Omega_n \subset I$ resulting in the measurements $\{f^{(n)}(i) : i \in \Omega_n\}$. Here $f^{(n)}(i) =< A^n f, e_i >$.

The measurements $\{f^{(0)}(i) : i \in \Omega_0\}$ that we have from our original signal $f = f^{(0)}$ will contain in general insufficient information to recover $f$. In other words, $f$ is undersampled. So we will need some extra information from the iterations of $f$ by the operator $A$: $\{f^{(n)}(i) = A^n f(i) : i \in \Omega_n\}$. Again, for each $n$, the measurements $\{f^{(n)}(i) : i \in \Omega_n\}$ that we have by sampling our signals $A^n f$ at $\Omega_n$ are insufficient to recover $A^n f$ in general.

Several questions arise. Will the combined measurements $\{f^{(n)}(i) : i \in \Omega_n\}$ contain in general all the information needed to recover $f$ (and hence $A^n f$)? How many iterations $L$ will we need (i.e., $n = 1, \dots, L$) to recover the original signal? What are the right "spatial" sampling sets $\Omega_n$ we need to choose in order to recover $f$? In what way all these questions depend on the operator $A$?

The goal of this paper is to answer these questions and understand completely this problem that we can formulate as:

Let $A$ be the evolution operator acting in $\ell^2(I)$, $\Omega \subset I$ be a fixed set of locations, and $\{l_i : i \in \Omega\}$ where $l_i$ is a positive integer or $+\infty$.

**Problem 1.1.** Find conditions on $A, \Omega$ and $\{l_i : i \in \Omega\}$ such that any vector $f \in \ell^2(I)$ can be recovered from the samples $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way.

Note that, in Problem 1.1, we allow $l_i$ to be finite or infinite. Note also that, Problem 1.1 is not the most general problem since the way it is stated implies that $\Omega = \Omega_0$ and $\Omega_n = \{i \in \Omega_0 : l_i \geq n\}$. Thus, an underlying assumption is that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 0$. For each $i \in \Omega$, let $S_i$ be the operator from $\mathcal{H} = \ell^2(I)$ to $\mathcal{H}_i = \ell^2(\{0, \dots, l_i\})$, defined by $S_i f = (A^j f(i))_{j=0,\dots,l_i}$ and define $S$ to be the operator $S = S_0 \oplus S_1 \oplus \dots$

Then $f$ can be recovered from $Y = \{f(i), Af(i), \dots, A^{l_i} f(i) : i \in \Omega\}$ in a stable way if and only if there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_2^2 \leq \|\mathcal{S} f\|_2^2 = \sum_{i \in \Omega} \|S_i f\|_2^2 \leq c_2 \|f\|_2^2. \tag{1}$$

Using the standard basis $\{e_i\}$ for $\ell^2(I)$, we obtain from (1) that

$$c_1 \|f\|_2^2 \leq \sum_{i \in \Omega} \sum_{j=0}^{l_i} |\langle f, A^{*j} e_i \rangle|^2 \leq c_2 \|f\|_2^2.$$

Thus we get

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