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Structure dependent sampling in compressed sensing: Theoretical guarantees for tight frames

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ABSTRACT

Many of the applications of compressed sensing have been based on variable density sampling, where certain sections of the sampling coefficients are sampled more densely. Furthermore, it has been observed that these sampling schemes are dependent not only on sparsity but also on the sparsity structure of the underlying signal. This paper extends the result of Adcock, Hansen, Poon and Roman (arXiv:1302.0561, 2013) [2] to the case where the sparsifying system forms a tight frame. By dividing the sampling coefficients into levels, our main result will describe how the amount of subsampling in each level is determined by the *local coherences* between the sampling and sparsifying operators and the *localized level sparsities* – the sparsity in each level under the sparsifying operator.

1. Introduction

Over the past decades, much of the research in signal processing has been based on the assumption that natural signals can be sparsely represented. One of the achievements resulting from this realization was compressed sensing, which made it possible to recover a sparse signal from very few non-adaptive linear measurements. Compressed sensing is typically modeled as follows. Given an unknown vector $x \in \mathbb{C}^N$ and a measurement device represented by a matrix V, one aims to recover x from a highly incomplete set of measurements by solving

$$R(x, \Omega) \in \operatorname*{argmin}_{z \in \mathbb{C}^N} \|Dz\|_{\ell^1} \text{ subject to } P_\Omega V z = P_\Omega V x, \tag{1.1}$$

where Ω indexes the given measurements, P_Ω is a projection matrix which restricts a vector to its coefficients indexed by Ω and D is a sparsifying matrix under which Dx is assumed to be sparse. Typical results in compressed sensing describe how under certain conditions, one can guarantee recovery when the number of measurements $|\Omega|$ scales up to a log factor linearly with sparsity [9,8,7].

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A large part of the theoretical development of compressed sensing has revolved around the construction of random sampling matrices (such as matrices constructed from random Gaussian ensembles) where the choice of the samples is completely independent of the sparsifying system [16,36,38,43]. The use of overcomplete dictionaries in compressed sensing has also been studied in works such as [6,20,28], but again, recovery guarantees were obtained only for randomized sampling matrices or subsampled structured matrices with randomized column signs. However, in the majority of applications where compressed sensing has been of interest, one is concerned with the recovery of a signal from structured measurements, without the possibility of first randomizing the underlying signal. For example, the measurements in magnetic resonance imaging (MRI) are modeled via the Fourier transform, while the measurements in radio interferometry are modeled via the Radon transform. In these cases, how one can achieve subsampling is *highly dependent* on the sparsifying transform. To explain this statement, we recall some results of compressed sensing on the recovery of a vector of length N from its discrete Fourier coefficients under various sparsifying transforms.

(1) If the underlying vector is s-sparse in its canonical basis, then one can guarantee perfect recovery from $\mathcal{O}(s \log N)$ Fourier coefficients drawn uniformly at random [8].
(2) If the underlying vector is s-sparse with respect to its total variation [8], then $\mathcal{O}(s \log N)$ Fourier coefficients drawn uniformly at random will again guarantee perfect recovery, however, in the presence of noise and approximate sparsity, then one can obtain superior error bounds with sampling strategies which sample more densely at low frequency coefficients instead [34].
(3) If the underlying vector is s-sparse with respect to some wavelet basis, then it is impossible to guarantee recovery from $\mathcal{O}(s \log N)$ samples from sampling uniformly at random. This is a phenomenon which has been observed since the early days of compressed sensing and there has been extensive investigations into how subsampling is still achievable by sampling more densely at low frequencies [33,31,39,42,35]. These approaches were often referred to as variable density sampling and theoretical guarantees for these approaches were recently derived in [29] and [2].

More generally, whether one can sample uniformly at random depends on whether the sampling and sparsifying matrices are sufficiently incoherent. In the absence of incoherence (as is the case in (3) above), how one should choose Ω in (1.1) becomes a far more delicate issue. To explain the use of compressed sensing in this case, a theoretical framework was developed in [2] on the basis of three new principles: multilevel sampling, asymptotic incoherence and asymptotic sparsity. By modelling a nonuniform sampling strategies via multilevel sampling, the need for dense sampling at low frequencies in (3) is due to the following two reasons.

(i) The high correspondence between Fourier and wavelet bases at low Fourier frequencies and low wavelet scales, but the low correspondence at high Fourier frequencies and high wavelet scales (asymptotic incoherence).
(ii) Typical signals or images exhibit distinctive sparsity patterns in their wavelet coefficients, and become increasingly sparse at higher wavelet scales (asymptotic sparsity).

In contrast to the large body of results in compressed sensing where the strategy is based on sparsity alone, the results of [2] demonstrated that one of the driving forces behind the success of variable density sampling strategies is their correspondence to the sparsity structure of the underlying signals of interest. These new principles provide a framework under which one can understand how to exploit both the sparsity structure of the underlying signal, and the correspondences between the sampling and sparsifying systems to devise optimal subsampling strategies [41,37].

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