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Letter to the Editor

An inequality for a periodic uncertainty constant

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An inequality refining the lower bound for a periodic (Breitenberger) uncertainty constant is proved for a wide class of functions. A connection of uncertainty constants for periodic and non-periodic functions is extended to this class. A particular minimization problem for a non-periodic (Heisenberg) uncertainty constant is studied.

1. Introduction

The Breitenberger uncertainty constant (UC) is commonly used as a measure of localization for periodic functions. It was introduced in 1985 by Breitenberger in [3]. It can be derived from a general operator “position-momentum” approach as it is discussed in [4]. The Breitenberger UC has a deep connection with the classical Heisenberg UC, which characterizes localization of functions on the real line. There exists a universal lower bound for both UCs (the uncertainty principle). It equals 1/2 (see chosen normalization in Sec. 2). It is well known that the least value is attained on the Gaussian function in the real line case and there is no such function in the periodic case. At the same time, in [2] Battle proves a number of inequalities specifying the lower bound of the Heisenberg UC for wavelets. In particular, it is proved that if a wavelet $\psi^0 \in L_2(\mathbb{R})$ has a zero frequency center $c(\widehat{\psi^0}) := \int_{\mathbb{R}} \xi |\widehat{\psi^0}(\xi)|^2 \, d\xi / (\int_{\mathbb{R}} |\widehat{\psi^0}(\xi)|^2 \, d\xi) = 0$, then the Heisenberg UC is greater or equal to 3/2 (see [2, Theorem 1.4]).

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The main contribution of this paper is an inequality refining the lower bound of the Breitenberger UC for a wide class of sequences of periodic functions (Theorem 5). This result is somewhat analogous to Battle's result mentioned above. Given a sequence of periodic functions ψ_j, $j \in \mathbb{Z}_+$, the conditions $|(\psi_j', \psi_j)| \leq C\|\psi_j\|^2$ and $\lim_{j\to\infty} q_j\widehat{\psi_j}(k)/\|\psi_j\| = 0$ (see (9) and (7) in Theorem 5) correspond to a zero frequency center $c(\widehat{\psi^0}) = 0$ and the wavelet admissibility condition $\widehat{\psi^0}(0) = 0$ respectively. The rest of restrictions (8), (10)–(12) in Theorem 5 mean some "regularity" of the sequence ψ_j. In [18], the following formula connecting UCs for periodic (UC_B) and non-periodic (UC_H) functions is obtained $\lim_{j\to\infty} UC_B(\psi_j^p) = UC_H(\psi_0)$, where $\psi_j^p(x) := 2^{j/2}\sum_{n\in\mathbb{Z}} \psi^0(2^j(x+2\pi n))$, $j \in \mathbb{Z}_+$. In Step 3 of the proof of Theorem 5, we generalize this formula and suggest a new proof of this fact. In Remarks 1–3, we discuss which classes of periodic wavelet sequences satisfy the conditions of Theorem 5. We also study one particular minimization problem for the Heisenberg UC connected with Battle's result mentioned above (Theorem 6). If the result of Theorem 6 had been wrong, it would have been possible to give another proof of Theorem 5.

While there are sufficiently many results specifying the lower and the upper bounds of the Heisenberg UC [1,2,4,5,7,9–12,15] and the upper bound of the Breitenberger UC [13,14,17,19,20], to our knowledge, there are no actually results concerning to an estimation of the lower bound for the Breitenberger UC in the literature.

This work also has the following motivation. In [13], a family of periodic Parseval wavelet frames is constructed. The family has optimal time–frequency localization (the Breitenberger UC tends to 1/2) with respect to a family parameter, and it has the best currently known localization (the Breitenberger UC tends to 3/2) with respect to a multiresolution analysis parameter. In [13], the conjecture was formulated: the Breitenberger UC is greater than 3/2 for any periodic wavelet sequence $(\psi_j)_{j\in\mathbb{Z}_+}$ such that $(\psi_j', \psi_j)_{L_{2,2\pi}} = 0$. Theorem 5 of this paper proves the conjecture for a wide class of sequences of periodic functions under a milder restriction $(\psi_j', \psi_j)_{L_{2,2\pi}} \leq C\|\psi_j\|^2_{L_{2,2\pi}}$. So the family constructed in [13] has optimal localization with respect to both parameters within the class of functions considered in Theorem 5.

2. Notations and auxiliary results

Let $L_{2,2\pi}$ be the space of all 2π-periodic square-integrable complex-valued functions, with inner product $(\cdot,\cdot)$ given by $(f, g) := (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$ for any $f, g \in L_{2,2\pi}$, and norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$. The Fourier series of a function $f \in L_{2,2\pi}$ is defined by $\sum_{k\in\mathbb{Z}} \widehat{f}(k)\mathrm{e}^{\mathrm{i}kx}$, where its Fourier coefficient is defined by $\widehat{f}(k) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$.

Let $L_2(\mathbb{R})$ be the space of all square-integrable complex-valued functions, with inner product $(\cdot,\cdot)$ given by $(f, g) := (2\pi)^{-1}\int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$ for any $f, g \in L_2(\mathbb{R})$, and norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$. The Fourier transform of a function $f \in L_2(\mathbb{R})$ is defined by $\widehat{f}(\xi) := (2\pi)^{-1}\int_{\mathbb{R}} f(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x$.

Let us recall the definitions of the UCs and the uncertainty principles.

Definition 1. (See [8].) `The Heisenberg UC` of $f \in L_2(\mathbb{R})$ is the functional $UC_H(f) := \Delta(f)\Delta(\widehat{f})$ such that

$$\Delta^2(f) := \frac{1}{\|f\|^2}\int_{\mathbb{R}} \left(x - c(f)\right)^2 \left|f(x)\right|^2 dx = \frac{\|(\cdot - c(f))f\|^2}{\|f\|^2},$$

$$c(f) := \frac{1}{\|f\|^2}\int_{\mathbb{R}} x\left|f(x)\right|^2 dx = \frac{(\cdot f, f)}{\|f\|^2},$$

where $\Delta(f)$, $\Delta(\widehat{f})$, $c(f)$, and $c(\widehat{f})$ are called `time variance, frequency variance, time centre,` and `frequency centre` respectively. By $\cdot f$, we denote the function that takes each x to $xf(x)$.

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