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A Laplace operator on complex Finsler manifolds

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we give the Laplace operator by defining a global inner product of (p, q) differential forms on strongly pseudoconvex compact complex Finsler manifolds, which can be regarded as an extension of that on Hermitian manifolds. Moreover, we derive the local coordinate expression of the Laplace operator. Finally, we prove that the Laplace operator is an elliptic self-adjoint operator and Hodge decomposition theorem holds.

It is well known that a Laplace operator plays an important role in the theory of harmonic integral and Bochner technique both in Riemannian and Kähler manifolds [5–7,21,15,10,19]. In the past two decades, under the initiation of S. S. Chern, the global differential geometry of real and complex Finsler manifolds has gained a great development [3,1]. The Laplace operator also makes sense in Finsler cases [2,4,11,20].

There are some results for complex Finsler manifolds. C. P. Zhong and T. D. Zhong [22] gave a Hodge–Laplace operator on a strongly pseudoconvex compact complex Finsler manifold M by pulling the (p, q) differential forms on M back to the projectivized tangent bundle $\mathbb{P}TM$ and then using the natural Hermitian inner product on $\mathbb{P}TM$ to obtain a global inner on M. And then they [23] obtained the Hodge decomposition theorem on strongly Kähler Finsler manifolds. J. X. Xiao, T. D. Zhong and C. H. Qiu [16,17] studied the Bochner and Bochner–Kodaira techniques in Kähler Finsler manifolds, moreover they [18] gave the explicit expression for the Laplace operator on the holomorphic tangent bundle of a strongly pseudoconvex complex Finsler manifold by using the Chern Finsler connection. J. L. Li, C. H. Qiu and T. D. Zhong [8] defined a natural projection of complex horizontal Laplacian on the projectivized tangent

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bundle $\mathbb{P}TM$, it is independent of the fiber coordinate, and proved the Hodge theorem for the natural projection of complex horizontal Laplacian on M. And then they [9] proved the Hodge decomposition theorem for the Hodge–Laplace operator on strongly pseudoconvex compact complex Finsler manifolds.

In this paper, by integration over fibers $B_z^{1,0}M$ of the sphere bundle $B^{1,0}M = \{v \in \tilde{M} | F(z,v) < 1\}$ of a given strongly pseudoconvex compact complex Finsler manifold (M, F), we are able to define a global inner product of the differential forms of type (p, q) on the base manifold M, and obtain a Laplace operator. Moreover we derive the local coordinate expression of the Laplace operator. Furthermore, we prove that the Laplace operator is an elliptic self-adjoint operator and Hodge decomposition theorem holds.

1. Preliminaries

Let M be a complex manifold of dimensions n. Denote by $\pi : T^{1,0}M \to M$ the holomorphic tangent bundle of M. Note that $T^{1,0}M$ may be a non-compact complex manifold even M is compact. For a local complex coordinate system $z = (z^1, \cdots, z^n)$ on M, a holomorphic tangent vector v of M is written as

$$v = v^i \frac{\partial}{\partial z^i}. \tag{1.1}$$

Take $(z, v) = (z^1, \cdots, z^n, v^1, \cdots, v^n)$ as local holomorphic coordinate neighborhood of $T^{1,0}M$. Let $\tilde{M} = T^{1,0}M \backslash \{0\}$ be $T^{1,0}M$ without the zero section. Then $\{\partial_i, \dot{\partial}_j\}$ gives a local holomorphic frame field of the holomorphic tangent bundle $T^{1,0}\tilde{M}$ of $\tilde{M}$, where

$$\partial_i = \frac{\partial}{\partial z^i} \quad \text{and} \quad \dot{\partial}_j = \frac{\partial}{\partial v^j}. \tag{1.2}$$

A complex Finsler metric on a complex manifold M is a continuous function $F : T^{1,0}M \to \mathbb{R}^+ \cup \{0\}$ with the following properties [1]:

(i) F^2 is smooth on $\tilde{M}$;
(ii) $F(v) > 0$ for all $v \in \tilde{M}$;
(iii) $F(\zeta v) = |\zeta| F(v)$ for all $v \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

The pair (M, F) is called a complex Finsler manifold. A complex Finsler metric F is said to be strongly pseudoconvex if the Levi matrix $G = (G_{i\bar{j}})$ is positive definite on $\tilde{M}$, where $G_{i\bar{j}} = \dot{\partial}_i \dot{\partial}_{\bar{j}}(F^2)$, and the pair (M, F) is called a strongly pseudoconvex complex Finsler manifold.

Let $\tilde{\pi} : T^{1,0}\tilde{M} \to \tilde{M}$ denote the natural projection. The differential $d\pi : T^{\mathbb{C}}\tilde{M} \to T^{\mathbb{C}}M$ of $\pi : \tilde{M} \to M$ defines the vertical bundle $\mathcal{V}$ over $\tilde{M}$ by

$$\mathcal{V} = \mathrm{Ker} d\pi \cap T^{1,0}\tilde{M}, \tag{1.3}$$

which is the holomorphic vector bundle of rank n over $\tilde{M}$, $\{\dot{\partial}_i\}$ gives a local frame for $\mathcal{V}$. As is defined in [1], there is a horizontal bundle $\mathcal{H}$ associated with the Chern–Finsler connection D over $\tilde{M}$ such that $T^{1,0}\tilde{M} = \mathcal{H} \oplus \mathcal{V}$, and local frame for $\mathcal{H}$ is given by

$$\delta_j = \partial_j - \Gamma_j^i \dot{\partial}_i, \tag{1.4}$$

where $\Gamma_j^i = G^{\bar{l}i} G_{\bar{l};j}$ are the Christoffel symbols of the Chern–Finsler connection D associated to $\mathcal{H}$, $(G^{\bar{l}i}) = (G_{i\bar{l}})^{-1}$, $G_{\bar{l};j} = \dot{\partial}_{\bar{l}} \partial_j (F^2)$. Thus $\{\delta_j, \dot{\partial}_i\}$ gives a local frame for $T^{1,0}\tilde{M}$. Let $\{dz^j, \delta v^i\}$ be the dual frame for $T^{1,0*}\tilde{M}$, where $\delta v^i = dv^i + \Gamma_j^i dz^j$. The frame $\{\delta_j, \dot{\partial}_i\}$ and $\{dz^j, \delta v^i\}$ are called the adapted frame for $T^{1,0}\tilde{M}$ and $T^{1,0*}\tilde{M}$ respectively.