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Volume preserving non-homogeneous mean curvature flow in hyperbolic space

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ABSTRACT

We study a volume/area preserving curvature flow of hypersurfaces that are convex by horospheres in the hyperbolic space, with velocity given by a generic positive, increasing function of the mean curvature, not necessarily homogeneous. For this class of speeds we prove the exponential convergence to a geodesic sphere. The proof is inspired by [9] and is based on the preserving of the convexity by horospheres that allows to bound the inner and outer radii and to give uniform bounds on the curvature by maximum principle arguments. In order to deduce the exponential trend, we study the behaviour of a suitable ratio associated to the hypersurface that converges exponentially in time to the value associated to a geodesic sphere.

1. Introduction

Let $\mathbb{H}_a^{n+1}$ be the hyperbolic space of constant sectional curvature $-a^2 < 0$ and let us take a smooth oriented, compact and without boundary hypersurface $F_0 : \mathcal{M} \to \mathbb{H}_a^{n+1}$. We consider a family of maps $F : \mathcal{M} \times [0, T) \to \mathbb{H}_a^{n+1}$, evolving according the law:

$$\begin{cases} \partial_t F(x,t) = [-\phi(H(x,t)) + h(t)]\nu(x,t) \\ F(x,0) = F_0(x), \end{cases} \tag{1.1}$$

where:

- H and ν denote respectively the mean curvature and the outer unit normal vector of the evolving hypersurface $\mathcal{M}_t := F(\mathcal{M}, t)$.

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- $\phi : [0, +\infty) \to \mathbb{R}$ is a continuous function, C^2 differentiable in $(0, +\infty)$ such that
  - *i)* $\phi(\alpha) > 0, \quad \phi'(\alpha) > 0 \quad \forall \alpha > 0$;
  - *ii)* $\lim_{\alpha\to\infty} \phi(\alpha) = \infty$;
  - *iii)* $\lim_{\alpha\to\infty} \frac{\phi'(\alpha)\alpha^2}{\phi(\alpha)} = \infty$;
  - *iv)* $\phi''(\alpha)\alpha \geq -2\phi'(\alpha) \quad \forall \alpha > 0$.
- The function $h(t)$ is either defined as

$$h(t) := \frac{1}{A_t} \int\limits_{\mathcal{M}_t} \phi(H) d\mu \tag{1.2}$$

or as

$$h(t) := \frac{\int_{\mathcal{M}_t} H\phi(H)d\mu}{\int_{\mathcal{M}_t} H d\mu}, \tag{1.3}$$

where $A_t = \int_{\mathcal{M}_t} d\mu_t$ denotes the area of $\mathcal{M}_t$.

The choice of h is made in order to keep the volume enclosed by $\mathcal{M}_t$ constant in case (1.2), and in order to keep the area of $\mathcal{M}_t$ constant in case (1.3). Flows of this form are sometimes called *constrained* curvature flows, while the corresponding ones without the $h(t)$ term will be referred to as *standard* flows.

In this paper we restrict our attention to the class of $\hbar$-convex hypersurfaces, that turns out to be a good choice when the ambient manifold is the hyperbolic space. Roughly speaking, we will see that $\hbar$-convexity is strong enough to offset the negative curvature of the ambient manifold and to be preserved along the flow. The main result proved in this paper is the following.

Theorem 1.1. *Let $F_0 : \mathcal{M} \to \mathbb{H}^{n+1}_a$, with $n \geq 1$, be a smooth embedding of an oriented, compact n-dimensional manifold without boundary, such that $F_0(\mathcal{M})$ is $\hbar$-convex. Then the flow (1.1) with $h(t)$ given by (1.2) (resp. (1.3)) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. The solution is $\hbar$-convex for any time and converges smoothly and exponentially, as $t \to \infty$, to a geodesic sphere that encloses the same volume (resp. has the same area) as the initial datum $\mathcal{M}_0$.*

A similar flow was recently studied by the first author and Sinestrari in [4] for strictly convex hypersurfaces of the Euclidean space. Since convexity is weaker than $\hbar$-convexity, the authors in [4] needed a certain behaviour of the velocity and its derivative at zero. We do not require these hypotheses on ϕ and ϕ'. Then we recover from [4] a very large class of velocities, as linear combinations of powers with degree greater than zero, logarithms and exponentials. On the other hand, we get some extra examples given by functions with a behaviour at zero not admitted in [4].

Constrained curvature flows have been intensively studied in recent years. One of the first result in this subject is [19]. For an overview on curvature flows in the Euclidean ambient manifold, see for example [10,22]. Our main source of inspiration is the paper of Cabezas-Rivas and Miquel [9], that we generalize. Theorem 1.2 in [9] in fact is a particular case of our Theorem 1.1 when $\phi(H) = H$ and h is taken as in (1.2). After [9], the evolution of $\hbar$-convex hypersurfaces was explored in many contests. For example, in [17,24] some mixed volume preserving flows were considered, where the velocity is assumed to be a degree one homogeneous function of the principal curvatures. In those cases too the authors have the convergence to a geodesic sphere. Also there are some results on curvature flows in the hyperbolic space with velocity of degree greater then one, but the conditions required on the initial datum are stronger than $\hbar$-convexity, as in [12]. In our paper instead we obtain in particular the convergence to a geodesic sphere for the flow