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Compact pseudo-Riemannian homogeneous Einstein manifolds of low dimension ☆

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A B S T R A C T

Let M be pseudo-Riemannian homogeneous Einstein manifold of finite volume, and suppose a connected Lie group G acts transitively and isometrically on M. In this situation, the metric on M induces a bilinear form $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ of G which is nil-invariant, a property closely related to invariance. We study such spaces M in three important cases. First, we assume $\langle\cdot,\cdot\rangle$ is invariant, in which case the Einstein property requires that G is either solvable or semisimple. Next, we investigate the case where G is solvable. Here, M is compact and $M = G/\Gamma$ for a lattice Γ in G. We show that in dimensions less or equal to 7, compact quotients $M = G/\Gamma$ exist only for nilpotent groups G. We conjecture that this is true for any dimension. In fact, this holds if Schanuel's Conjecture on transcendental numbers is true. Finally, we consider semisimple Lie groups G, and find that M splits as a pseudo-Riemannian product of Einstein quotients for the compact and the non-compact factors of G.

1. Introduction and main results

Let M be a pseudo-Riemannian homogeneous manifold with finite volume, such that $M = G/H$, where G is a connected Lie group with a closed subgroup H, and G acts isometrically and almost effectively on M (meaning H contains no connected normal subgroup of G). We further assume that the pseudo-Riemannian metric g_M on M is an Einstein metric, that is, the Ricci tensor is a multiple of the metric, $\mathsf{Ric} = \lambda \mathsf{g}_M$, for some real constant λ. This work is motivated by questions on the existence of such spaces M, and the algebraic structure of the groups G.

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The pseudo-Riemannian metric g_M on M pulls back to a symmetric bilinear tensor on G, and thus induces a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{g}$ of G. We review some basic notions of scalar products on Lie algebras in Section 2.

The fact that M has finite volume implies that $\langle\cdot,\cdot\rangle$ is nil-invariant (Definition 2.2), a property closely related to invariance of $\langle\cdot,\cdot\rangle$ (see Baues and Globke [2]). It was shown by Baues and Globke [2, Theorem 1.2] that a nil-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on a finite-dimensional solvable Lie algebra must be invariant. The structure of solvable Lie algebras with invariant scalar product can be understood through a sequential reduction to lower-dimensional algebras of the same type. This process is briefly reviewed in Section 3.

In Section 4 we consider Lie groups with Einstein metrics. If the Einstein metric is bi-invariant, we can make an easy but very important observation:

Proposition 1.1. *If g is a bi-invariant Einstein metric on G, then*

(a) *either G is semisimple and g is a non-zero multiple of the Killing form,*
(b) *or G is solvable and its Killing form vanishes.*

Combining this proposition with results from [3] shows that for any pseudo-Riemannian homogeneous Einstein manifold $M = G/H$ of finite volume for a Lie group G whose Levi subgroup has no compact factors, G is either semisimple or solvable (Corollary 4.3), and the stabilizer H is a lattice in G.

We study pseudo-Riemannian homogeneous Einstein manifolds $M = G/H$ of finite volume and with G a connected solvable Lie group in Section 5. The stabilizer H is a lattice in G, and this implies that the induced bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ is in fact a scalar product. Moreover, $\langle\cdot,\cdot\rangle$ is invariant and therefore the Killing form of $\mathfrak{g}$ vanishes in order to satisfy the Einstein condition. This is trivially satisfied by any nilpotent Lie algebra, so that every compact pseudo-Riemannian nilmanifold is an Einstein manifold. For solvable Lie algebras that are not nilpotent, the Killing form does not vanish in general. By a metric Lie algebra we mean a Lie algebra with an invariant scalar product. We find the following constraints on the dimension of $\mathfrak{g}$:

Theorem 1.2. *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be a solvable metric Lie algebra with Einstein scalar product of Witt index s. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. If $\mathfrak{g}$ is not nilpotent, then* $\dim\mathfrak{g} \geq 6$, $\dim\mathfrak{n} \geq 5$, *and* $s \geq 2$. *Example 4.4 shows that these estimates are sharp.*

The existence of compact pseudo-Riemannian quotients of a solvable Lie group G requires the existence of a lattice Γ in G, and it is not clear whether such a lattice can exist under the condition that $\langle\cdot,\cdot\rangle$ is Einstein. We can rule this out for dimensions below 8:

Theorem 1.3. *Let M be a compact pseudo-Riemannian Einstein solvmanifold of dimension less or equal to* 7. *Then M is a nilmanifold.*

The proof of this theorem uses the Gelfond–Schneider Theorem from the theory of transcendental numbers to show that no lattice can exist. However, application of the Gelfond–Schneider Theorem seems limited to low dimensions. For arbitrary dimensions, the statement would hold if Schanuel's Conjecture on the algebraic independence of certain complex numbers is true. We therefore conjecture:

Conjecture 1.4. *Every compact pseudo-Riemannian Einstein solvmanifold is a nilmanifold.*

In Section 6 we consider pseudo-Riemannian homogeneous Einstein manifolds $M = G/H$ of finite volume, where G is a semisimple Lie group. We may assume that G is simply connected, so $G = K \times S$, where K