ARTICLE IN PRESS DIFGEO:1327

Differential Geometry and its Applications ••• (••••) •••–•••

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Improper affine spheres and the Hessian one equation ☆

Antonio Martínez, Francisco Milán *

Departamento de Geometría y Topología, Universidad de Granada, E-18071 Granada, Spain

ARTICLE INFO

Article history:
Received 14 October 2016
Received in revised form 19 January 2017
Available online xxxx
Communicated by E. Garcia Rio

MSC:
53A15

Keywords:
Hessian one equation
Improper affine spheres
Ribaucour transformations
Cauchy problem

ABSTRACT

Improper affine spheres have played an important role in the development of geometric methods for the study of the Hessian one equation. Here, we review most of the advances we have made in this direction during the last twenty years.

1. Introduction

Differential Geometry and Partial Differential Equations (PDEs) are related by a productive tie by means of which both theories well out. On the one hand, many results from the theory of hypersurfaces and submanifolds use, in a fundamental way, tools from PDEs theory. Among them, we can emphasize topics like maximum principles, regularity theorems, height estimates, asymptotic behavior at infinity, representation theorems, or results of existence and uniqueness with fixed boundary conditions.

On the other hand, many classic PDEs are linked to interesting geometric problems. Even more, apart from being source of inspiration in the search for interesting PDEs, the geometry allows, in many cases, to integrate these equations, to establish non-trivial properties of the solutions and to give superposition principles which determine new solutions in terms of already known solutions.

☆ Research partially supported by Ministerio de Educación y Ciencia Grant No. MTM2016-80313-P and Junta de Andalucía Grant No. FQM-325.

* Corresponding author.
E-mail addresses: amartine@ugr.es (A. Martínez), milan@ugr.es (F. Milán).

http://dx.doi.org/10.1016/j.difgeo.2017.02.001
0926-2245/© 2017 Elsevier B.V. All rights reserved.

One of the biggest contributions from geometry to PDEs theory are Monge–Ampère equations. Such equations are totally non-linear PDEs which model interesting geometric aspects related to the curvature, and its study has become a topic of great mathematical importance, see [15,23].

Among the most outstanding Monge–Ampère equations we can quote the Hessian one equation

$$u_{xx}u_{yy} - u_{xy}^2 = \varepsilon, \qquad \varepsilon \in \{-1, 1\}, \qquad (x, y) \in \Omega \subseteq \mathbb{R}^2. \tag{H_ϵ}$$

The equation (H_ϵ) has been studied from many perspectives by several authors and the situation changes completely if we take $\varepsilon = 1$ (elliptic case) or $\varepsilon = -1$ (hyperbolic case). When $\varepsilon = 1$, we know there exists a representation for the solutions in terms of holomorphic data and their most relevant global properties have been studied, such as the behavior of the ends, the flexibility or the existence of symmetric solutions (see [9,10]). In [3,11] the space of solutions of the exterior Dirichlet problem is endowed with a suitable structure and in [12], the space of solutions of the equation defined on $\mathbb{R}^2$ with a finite number of points removed was described, and endowed with the structure of a differential manifold. In [1] the Cauchy problem was solved, and this solution was used, among other things, to classify the solutions which have an isolated singularity. Thanks to classical subjects of the PDEs theory, such as the continuity method, it seems possible to extend some of the above results to more general classes of Monge–Ampère equations. However, the hyperbolic case is much more complicated and we can not expect classification results as in the definite case. For example, $u(x, y) = xy + g(x)$ is an entire solution for any real function g.

From a geometric point of view, the equation (H_ϵ) arises as the equation of improper affine spheres, that is, surfaces with parallel affine normals. Although Affine Differential Geometry has a long history whose origins date back to 1841 in a work by Transon on the normal affine of a curve, it has been during the last thirty years when this theory has experienced a remarkable development. In its research geometric, analytic and complex techniques mix and in order to understand the geometry of this important family of surfaces deeply, the global rigidity has been weakened in different ways. On the one hand, in [9,10], using methods of complex analysis, their behavior at the infinity has been described noting that there exists a closely relationship between their ends and the ones of a minimal surface with finite total curvature. On the other hand and concerning with (H_ϵ), a geometric theory of smooth maps with singularities (improper affine maps) has been developed opening an interesting range of global examples. In most of the cases the singular set determines the surface and, generically, the singularities are cuspidal edges and swallowtails, see [8,13,17].

In very recent works, [18–21] we have solved the problem of finding all indefinite improper affine spheres passing through a given regular curve of $\mathbb{R}^3$ with a prescribed affine co-normal vector field along this curve; the problem is well-posed when the initial data are non-characteristic and the uniqueness of the solution can fail at characteristic directions. We also have learnt how to obtain easily improper affine maps with a prescribed singular set, how to construct global examples with the desired singularities and how to use the classical theory of Ribaucour transformations to obtain new solutions of the elliptic Hessian one equation.

Here, we want to make a short survey with some of the above mentioned advances in the geometric study of (H_ϵ).

2. A complex (split-complex) representation formula

If $u : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a solution of (H_ϵ), then its graph

$$\psi = \{(x, y, u(x, y)) : (x, y) \in \Omega\}$$

is an improper affine sphere in $\mathbb{R}^3$ with constant affine normal $\xi = (0, 0, 1)$, affine metric h,

$$h := u_{xx}\, dx^2 + u_{yy}\, dy^2 + 2u_{xy}\, dxdy, \tag{2}$$

and affine conormal N,

Please cite this article in press as: A. Martínez, F. Milán, Improper affine spheres and the Hessian one equation, Differ. Geom. Appl. (2017), http://dx.doi.org/10.1016/j.difgeo.2017.02.001