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Quantum potential in covariant quantum mechanics

Josef Janyška [a,*,1], Marco Modugno [b]

[a] *Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic*
[b] *Department of Mathematics and Informatics "U. Dini", University of Florence, Via S. Marta 3, 50139 Florence, Italy*

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A B S T R A C T

We discuss several features of the classical quantum potential appearing in Covariant Quantum Mechanics.
In particular, we compare the "observed potential" $A[K, G, o]$ of the joined spacetime connection K with the potential $A^{\uparrow}$ of the cosymplectic phase 2-form $\Omega[K, G]$ and with the potential $A^{\uparrow}$ of the upper quantum connection $\text{Ч}^{\uparrow}$.
Moreover, we discuss the distinguished observer $o[\Psi]$ and the distinguished timelike potential $A[\Psi]$ associated with a non-vanishing quantum section Ψ.
We show that the above objects play a natural role in the context of the kinetic quantum momentum $\mathrm{Q}[\Psi]$, of the quantum probability current $\mathrm{J}[\Psi]$, of the Schrödinger operator $\mathrm{S}[\Psi]$ and of the classical fluid associated with a non-vanishing quantum section Ψ.

0. Introduction

Starting from E. Cartan [8], there have been proposed several formulations of Quantum Mechanics in a curved spacetime with absolute time (see, for instance, [2–4,12,17–23,28,37–41,52,54,59] and citations therein).

* Corresponding author.
E-mail addresses: janyska@math.muni.cz (J. Janyška), marco.modugno@unifi.it (M. Modugno).
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Covariant Quantum Mechanics is an approach to Quantum Mechanics in a curved spacetime fibred over time and equipped with absolute time and a riemannian metric on its fibres, aimed at implementing several features of General Relativity in this riemannian framework. This formulation started some years ago [26] and has been further developed by several papers (see, for instance, [6,7,25,31,44–47,49,50,56,57] and citations therein).

Several ideas and methods are typical features of Covariant Quantum Mechanics. For instance, we consider as phase space the 1st jet space $J_1\boldsymbol{E}$, we couple the gravitational field $K^\natural$ and the electromagnetic field F into a *joined* spacetime connection, which yields several joined objects of the phase space, such as the *cosymplectic* 2-form Ω, which plays a fundamental role in classical and quantum mechanics. Moreover, we introduce the special phase functions and their Lie bracket. In the quantum theory, we introduce a complex line bundle $\boldsymbol{Q}$ over spacetime and an *upper quantum connection* $Ч^\uparrow$, which is hermitian and "reducible" and whose curvature is proportional to Ω. All main further quantum objects are derived in a natural way from this connection, by means of a "criterion of projectability", which allows us to get rid of observers, in view of the covariance of the theory. The quantum operators are achieved via the classification of hermitian quantum vector fields and their Lie algebra isomorphism with the special phase functions.

Scales. We deal with units of measurement on the same footing of gauges, observers and coordinates. So, in order to make our theory explicitly independent of "*units of measurement*", we use the notion of "*spaces of scales*" [32].

We define a *positive space* to be a semi-vector space $\mathbb{U}$ on the semi-field $\mathbb{R}^+$, such that the scalar product $\cdot : \mathbb{R}^+ \times \mathbb{U} \to \mathbb{U}$ is a left free and transitive action of the group $(\mathbb{R}^+, \cdot)$ on $\mathbb{U}$. We can define in a natural way the tensor product $\mathbb{U} \otimes \mathbb{U}'$ of two positive spaces, the rational powers $\mathbb{U}^{m/n}$ of a positive space and the dual $\mathbb{U}^*$ of a positive space. We make a natural identification $\mathbb{U}^* \simeq \mathbb{U}^{-1}$. Moreover, we can define in a natural way the tensor product $\mathbb{U} \otimes \boldsymbol{V}$ of a positive space with a vector space; indeed, it turns out to be a vector space.

We consider the following *basic positive spaces*: 1) the space $\mathbb{T}$ of *time intervals*, 2) the space $\mathbb{L}$ of *lengths*, 3) the space $\mathbb{M}$ of *masses*. Then, we define a *space of scales* to be any tensor product of rational powers of the above positive spaces.

We consider the *Planck constant* $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ as a "*universal scale*". Moreover, we will consider a *mass* $m \in \mathbb{M}$ and *charge* $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}$. We denote a *time unit of measurement* and its dual, respectively, by $u_0 \in \mathbb{T}$ and $u^0 \in \mathbb{T}^* \simeq \mathbb{T}^{-1}$.

1. Setting of the classical theory

We start by summarising some achievements of Covariant Classical Mechanics.

1.1. Spacetime

We consider *time* to be an oriented 1-dimensional affine space $\boldsymbol{T}$, associated with the vector space $\mathbb{T} \otimes \mathbb{R}$, and *spacetime* to be an oriented 4-dimensional manifold $\boldsymbol{E}$ equipped with a *time fibring*

$$t : \boldsymbol{E} \to \boldsymbol{T} .$$

The time fibring yields the distinguished *time form* $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$.

A *motion* is defined to be a section $s : \boldsymbol{T} \to \boldsymbol{E}$.

We shall refer to *spacetime charts* $(x^\lambda) \equiv (x^0, x^i)$, defined as charts of the manifold $\boldsymbol{E}$, which are adapted to the time fibring, the affine structure of $\boldsymbol{T}$ and the orientation of $\boldsymbol{E}$ and $\boldsymbol{T}$. Every spacetime chart (x^λ) yields a *time scale* $u_0 \in \mathbb{T}$. We shall denote the associated bases of vector fields and forms by $(\partial_\lambda) \equiv (\partial_0, \partial_i)$