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A class of nilpotent Lie algebras admitting a compact subgroup of automorphisms ☆

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A B S T R A C T

The realification of the $(2n+1)$-dimensional complex Heisenberg Lie algebra is a $(4n+2)$-dimensional real nilpotent Lie algebra with a 2-dimensional commutator ideal coinciding with the centre, and admitting the compact algebra $\mathfrak{sp}(n)$ of derivations. We investigate, in general, whether a real nilpotent Lie algebra with 2-dimensional commutator ideal coinciding with the centre admits a compact Lie algebra of derivations. This also gives us the occasion to revisit a series of classic results, with the expressed aim of attracting the interest of a broader audience.

1. Introduction

Metabelian Lie algebras $\mathfrak{h} = V \oplus \langle x, y\rangle$ of dimension $(n+2)$ defined by a pair of alternating forms F_1, F_2 on the n-dimensional vector space V, putting, for any $v, w \in V$, $[v, w] = F_1(v, w)x + F_2(v, w)y$, are called nilpotent Lie algebras of type $\{n, 2\}$. The *type* $\{d_1, \dots, d_c\}$ of a nilpotent Lie algebra $\mathfrak{g}$ with descending central series $\mathfrak{g}^{(i)} = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$ is defined, according to the literature beginning with Vergne [20], by the integers $d_i = \dim \frac{\mathfrak{g}^{(i-1)}}{\mathfrak{g}^{(i)}}$. Nilpotent (real or complex) Lie algebras of type $\{n, 2\}$ have been classified firstly by Gauger [11], applying the canonical reduction of the pair F_1, F_2, but, according to results of Belitskii, Lipyanski, and Sergeichuk [3], it is not possible to carry this argument further. On the contrary, it seems possible to broaden these families of Lie algebras by considering their derivations, this has been done in [9]

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extending real Lie algebras of type $\{n, 2\}$ by a single compact derivation. We mention that also nilpotent Lie algebras of type $\{n, 1, 1\}$ can be explicitly described (cf. [2]), and derivations of a nilpotent Lie algebra of type $\{2n, 1, 1\}$ are determined in [1].

In the present paper, we are interested in the following problem. The realification of the $(2n+1)$-dimensional complex Heisenberg Lie algebra $\widehat{\mathfrak{h}}$ is a $(4n+2)$-dimensional real Lie algebra $\mathfrak{h}$ of type $\{4n, 2\}$, admitting the compact algebra $\mathfrak{sp}(n)$ of derivations. We ask, in general, whether a real Lie algebra $\mathfrak{h}$ of type $\{2n, 2\}$ admits a compact Lie algebra of derivations, that is, the Lie algebra of a compact Lie group, a question arising in the study of the isometry groups of homogeneous nilmanifolds. In fact, the realifications $\mathfrak{h} = V \oplus \langle x, y\rangle$ of the complex Heisenberg algebras $\widehat{\mathfrak{h}}$ are the only H-type algebras with two-dimensional centre (cf. [19], Section 5, p. 3252), where an H-type algebra is a nilpotent Lie algebra of class 2 with an inner product such that the operator $J : \mathfrak{h} \longrightarrow \mathrm{End}(V)$, $z \mapsto J_z$, defined by $\langle J_z v, w\rangle = \langle z, [v, w]\rangle$, fulfills $J_z^2 = -\langle z, z\rangle \mathrm{id}_V$.

Notice that a non-commutative simple compact Lie algebra of derivations of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{2n, 2\}$ must induce the null map on the two-dimensional commutator ideal $\mathfrak{h}'$, because a non-commutative simple compact Lie algebra cannot have a two-dimensional representation.

Generally speaking, for the compact Lie algebra $\mathfrak{g}$ of a compact Lie group G, the opposite of the Killing form induces on $\mathfrak{g}$ an $\mathrm{Ad}(G)$-invariant inner product, and, up to scalar multiplication, this is the unique $\mathrm{Ad}(G)$-invariant inner product. With respect to this inner product, $\mathrm{Ad}(G)$ acts by orthogonal transformations of $\mathrm{SO}(\mathfrak{g})$ and $\mathrm{ad}(\mathfrak{g})$ acts by skew-symmetric matrices of $\mathfrak{so}(\mathfrak{g})$. Therefore, a compact Lie algebra can be embedded into $\mathfrak{so}(\mathfrak{g})$, a simpler and stronger version of Ado's theorem.

In the context of (real) nilpotent Lie algebras with low-dimensional commutator ideals, the situation is as follows. Let $\mathfrak{g}$ be a nilpotent Lie algebra with commutator ideal $\mathfrak{g}'$ and centre $\mathfrak{z}$.

- If $\dim \mathfrak{g}' = 0$, then $\mathfrak{g}$ is Abelian and the maximal compact subgroup is $\mathrm{O}(n)$.
- If $\dim \mathfrak{g}' = 1$ and $\dim \mathfrak{z} = 1$, then $\mathfrak{g}$ is the $(2n+1)$-dimensional Heisenberg algebra $\mathfrak{h}_{2n+1}$ and the maximal compact subgroup is $\mathrm{U}(n) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{O}(2n)$ (cf. [4]). Note that if instead $\dim \mathfrak{z} = \ell + 1$ then $\mathfrak{g} \cong \mathfrak{h}_{2n+1} \oplus \mathbb{R}^\ell$.
- If $\dim \mathfrak{g}' = 2$ and $\dim \mathfrak{z} = 1$, then $\mathfrak{g}$ is a uniquely determined by its dimension (cf. [2]); the $(2n+4)$- and $(2n+5)$-dimensional groups turn out to have maximal compact subgroup $\mathrm{U}(n) = \mathrm{Sp}(2n, \mathbb{R}) \cap \mathrm{O}(2n)$. Note again that if instead $\dim \mathfrak{z} = \ell + 1$, then we simply have a trivial Abelian extension of the case already discussed.
- Finally, if $\mathfrak{g}' = \mathfrak{z}$ and $\dim \mathfrak{g}' = 2$, we have the case considered in this paper. This is arguably the first interesting case (in this list) as there exist Lie algebras of the same dimension of this kind with different maximal compact subgroups of automorphisms (see Examples 1, 2).

We give remarkable examples for nilmanifolds M such that the group of isometries of M contains the compact group $\mathrm{SO}_2(\mathbb{R})$.

In the present Introduction, we introduce the Heisenberg Lie algebra in the contexts of algebra, complex analysis and quantum mechanics. Although not directly connected, these results appear in all introductory books in these fields. Nevertheless, they serve us to emphasize the non-trivial rôle of compact derivations of nilpotent Lie algebras.

1.1. Heisenberg algebra, Wirtinger derivatives, and Weyl algebra

Heisenberg Lie algebras are the most elementary non-Abelian Lie algebras. Such a Lie algebra $\mathfrak{h} = V \oplus \langle u\rangle$ has dimension $(2n+1)$ and is defined by a non-degenerate alternating form F on the $2n$-dimensional subspace V, putting $[v, w] = F(v, w)u$, for any $v, w \in V$. The choice of a symplectic basis $\{p_1, q_1, \dots, p_n, q_n\}$ of V allows one to write