ARTICLE IN PRESS
DIFGEO:1368

Differential Geometry and its Applications ••• (••••) •••–•••

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Construction of discrete constant mean curvature surfaces in Riemannian spaceforms and applications

Yuta Ogata [a], Masashi Yasumoto [b,*]

[a] Department of Integrated Arts and Science, National Institute of Technology, Okinawa College, Henoko 905, Nago-shi, 905-2192, Japan
[b] Osaka City University Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

A R T I C L E   I N F O

Article history:
Received 20 August 2016
Received in revised form 29 March 2017
Available online xxxx
Communicated by F. Pedit

MSC:
53A10
52C99

Keywords:
Discrete differential geometry
Surface theory
Integrable systems
Constant Gaussian curvature surfaces
Singularities

A B S T R A C T

In this paper we give a construction for discrete constant mean curvature surfaces in Riemannian spaceforms in terms of integrable systems techniques, which we call the discrete DPW method for discrete constant mean curvature surfaces. Using this construction, we give several examples, and analyze singularities of the parallel constant Gaussian curvature surfaces.

1. Introduction

In the smooth (or continuous) case, surfaces governed by some integrable equation, like constant negative Gaussian curvature surfaces and non-zero constant mean curvature (for short, CMC) surfaces, have been well studied. When studying such surfaces, it is useful to describe 2×2 matrix representations and matrix-valued partial differential equations called Lax pairs. In particular, applying matrix-splitting theorems, Dorfmeister, Pedit, Wu [9] established the generalized Weierstrass representation for smooth CMC surfaces in Euclidean 3-space $\mathbb{R}^3$ (regarding the cases of smooth CMC surfaces in spherical 3-space $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$, see [2,10] for example). This representation is now called the *DPW method* for smooth CMC surfaces.

* Corresponding author.
E-mail addresses: y.ogata@okinawa-ct.ac.jp (Y. Ogata), yasumoto@sci.osaka-cu.ac.jp (M. Yasumoto).
URL: https://sites.google.com/site/homepageofmasashiyasumoto/home (M. Yasumoto).

http://dx.doi.org/10.1016/j.difgeo.2017.04.010
0926-2245/

Stepping away from smooth surface theory, there has been recent progress on discrete surface theory. In the last three decades, using integrable systems techniques, discrete surface theory has been developed. Burstall, Hertrich-Jeromin, Rossman, Santos [8] described discrete CMC surfaces in any 3-dimensional Riemannian spaceform and gave several new examples of discrete CMC surfaces, and Bobenko, Hertrich-Jeromin, Lukyanenko [3] gave a curvature theory for discrete surfaces in Riemannian spaceforms (see also [7]). Due to these works, we are able to treat discrete surfaces in 3-dimensional Riemannian spaceforms. In particular, constructing discrete CMC surfaces is one of the central topics in the study of discrete surface theory. In fact, Bobenko and Pinkall [1] introduced a Weierstrass representation for discrete minimal surfaces in $\mathbb{R}^3$, and Hertrich-Jeromin [11] derived a Weierstrass-type representation for discrete CMC 1 surfaces in $\mathbb{H}^3$.

Bobenko, Pinkall [4] described Lax pairs for discrete CMC surfaces in $\mathbb{R}^3$ and gave a Cauchy problem for them (see also [12]). Applying matrix-splitting formulae (see also Propositions 4.1, 4.2 here), Hoffmann [13] gave a construction for discrete non-zero CMC surfaces in $\mathbb{R}^3$. This method is called the *discrete DPW method* for discrete CMC surfaces in $\mathbb{R}^3$. On the other hand, although discrete CMC surfaces became treatable recently, the discrete DPW method for discrete CMC surfaces in other 3-dimensional Riemannian spaceforms had not yet been considered.

In this paper, we give the discrete DPW method for discrete CMC surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$, which is a generalization of the work by Hoffmann [13], and give several examples. In the smooth case, we can choose a common Lax pair for $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$. Also in the discrete case, using the same Lax pair as in $\mathbb{R}^3$, we will show that we can construct discrete CMC surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ and that discrete CMC surfaces given by Lax pairs have mean curvatures (in the sense of [3] and [7]) that are constant. Our construction covers discrete isothermic surfaces in $\mathbb{S}^3$ with any constant mean curvature H, and the discrete isothermic surfaces in $\mathbb{H}^3$ with constant mean curvature H satisfying $|H| > 1$.

As an application, we will also construct discrete constant positive Gaussian curvature surfaces by taking parallel surfaces of discrete CMC surfaces, and look at their singularities. In the smooth case, constant Gaussian curvature surfaces generally have singularities (for example, see [15]), so it is natural to expect that discrete constant positive Gaussian curvature surfaces have certain configurations of singularities. Based on work by Rossman and the second author [19], we will analyze singularities of such discrete constant positive Gaussian curvature surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$.

2. The DPW method for smooth CMC surfaces

First we introduce construction of smooth CMC $H \neq 0$ surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ introduced in [9] (see also [10]), which is now called the DPW method. Throughout this paper, $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbb{S}^3 := \{Y \in \mathbb{R}^4 | \langle Y, Y \rangle = 1\}$, $\mathbb{R}^{3,1}$ denotes Minkowski 4-space with signature $(+++-)$, and $\mathbb{H}^3 := \{Y \in \mathbb{R}^{3,1} | \langle Y, Y \rangle = -1\}$.

Let Σ be a simply connected domain in the complex plane $\mathbb{C}$ with the usual complex coordinate $z = x+iy$, let $f : \Sigma \to \mathbb{R}^3$ be a conformal immersion satisfying $\langle f_x, f_x \rangle = \langle f_y, f_y \rangle = 4e^{2u}$, $\langle f_x, f_y \rangle = 0$ for some scalar function $u : \Sigma \to \mathbb{R}$, and let $N : \Sigma \to \mathbb{S}$ be its unit normal vector field. In this paper we identify $\mathbb{R}^4$ (resp. $\mathbb{R}^{3,1}$) with the unitary group $\left\{X \in M_{2\times 2} \,\middle|\, X \cdot \bar{X}^t = I\right\}$ (resp. another matrix group) as follows:

$$\mathbb{R}^4 \text{ (or, } \mathbb{R}^{3,1}) \ni x = (x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_4 + \nu \cdot x_3 & x_1 - ix_2 \\ -\epsilon \cdot (x_1 + ix_2) & x_4 - \nu \cdot x_3 \end{pmatrix}, \quad (1)$$

with $\nu = i, \epsilon = 1$ for $\mathbb{R}^4$ (resp. $\nu = 1, \epsilon = -1$ for $\mathbb{R}^{3,1}$). The metric becomes, under this identification,

$$\langle X, Y \rangle = \epsilon \cdot \frac{1}{2}\mathrm{trace}(X\sigma_2 Y^t \sigma_2).$$

Please cite this article in press as: Y. Ogata, M. Yasumoto, Construction of discrete constant mean curvature surfaces in Riemannian spaceforms and applications, Differ. Geom. Appl. (2017), http://dx.doi.org/10.1016/j.difgeo.2017.04.010