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Deformation of piecewise differentiable curves in constrained variational calculus

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ABSTRACT

A survey of the geometric tools involved in the study of constrained variational calculus is presented. The central issue is the characterization of the admissible deformations of piecewise differentiable sections of a fibre bundle $\mathcal{V}_{n+1} \to \mathbb{R}$, in the presence of arbitrary non-holonomic constraints. Asynchronous displacements of the corners are explicitly considered. The coordinate-independent representation of the variational equation and the associated concepts of *infinitesimal control* and *absolute time derivative* are reviewed. In the resulting algebraic environment, every admissible section is assigned a corresponding *abnormality index*, identified with the co-rank of a suitable linear map. Sections with vanishing index are called *normal*. A section is called *ordinary* if every solution of the variational equation vanishing at the endpoints is tangent to some finite deformation with fixed endpoints. The interplay between abnormality index and ordinariness — in particular the fact that every normal evolution is automatically an ordinary one — is discussed.

0. Introduction

Calculus of variations has a very old origin, dating back to the pioneering works of Euler, Lagrange and Weierstrass.[1] More recent contributions [11–22] have significantly improved the differential geometric approach to the subject.

In this paper we review some foundational aspects of constrained variational calculus. The discussion deals with *parameterized curves*, namely with sections of a fibre bundle $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, called the *event space*, the projection t possibly identified with the *absolute time* of Classical Mechanics. The constraints are accounted for by a submanifold $\mathcal{A}$ of the first jet bundle of $j_1(\mathcal{V}_{n+1})$. Every continuous, piecewise differentiable section

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[1] For a modern exposition of the classical theory, see e.g. [1–10].

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$\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ whose first jet extension $j_1(\gamma)$ factors through $\mathcal{A}$ is called an *admissible evolution*. The points of discontinuity of $j_1(\gamma)$ are called the *corners* of γ.

A basic task of constrained variational calculus is characterizing the extremals of a given action functional I among the class of admissible evolutions.

In this connection, the presence of kinetic constraints and the possible existence of corners raise some relevant questions: among others, the covariant characterization of the infinitesimal deformations, the geometrical interpretation of the concept of *normality* of a section, the relation between normality and deformability. A thorough analysis of these aspects may be found in [20].

The main results are reported in the following Sections. After a few preliminary remarks, the infinitesimal deformations of an admissible evolution γ are discussed via a revisitation of the *variational equation*. The central idea is the introduction of the concept of *infinitesimal control*, yielding a covariant characterization of the (infinite dimensional) vector space $\mathfrak{W}$ formed by the totality of admissible infinitesimal deformations.

In Section 2 the admissible evolutions are classified into *ordinary*, if every element of $\mathfrak{W}$ vanishing at the endpoints of γ is tangent to some finite deformations with fixed endpoints, and *exceptional* in the opposite case. Along the same guidelines, every admissible evolution is assigned a corresponding *abnormality index*, extending and expressing in geometrical terms the traditional attributes of normality and abnormality commonly found in the literature.

The interplay between abnormality index and ordinariness is eventually discussed in Subsection 2.2. The fact that all normal evolutions are automatically ordinary, proved by Hsu [12] in a linear context, is established in the case of arbitrary non-linear constraints and piecewise differentiable sections.

Section 3 provides arguments and examples that clarify some aspects of the concept of normality discussed in Sec. 2.1.

1. Overview of foregoing results

All definitions, conventions and results described in [20] will be freely used throughout. For convenience of the reader, a few basic aspects, especially relevant to the present discussion, are reported below.

1.1. Preliminaries

(i) Let $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$ denote a $(n+1)$-dimensional fibre bundle, locally referred to fibred coordinates $t, q^1, \ldots, q^n$ and called the *event space*.

Every section $\gamma\colon \mathbb{R} \to \mathcal{V}_{n+1}$ is interpreted as the evolution of an abstract system with a finite number of degrees of freedom.

The first jet bundle $j_1(\mathcal{V}_{n+1}) \xrightarrow{\pi} \mathcal{V}_{n+1}$ (with π denoting the natural projection), referred to jet coordinates $t, q^i, \dot{q}^i$, is called the *velocity space*. The jet-extension of a section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ is indicated by $j_1(\gamma)\colon \mathbb{R} \to j_1(\mathcal{V}_{n+1})$.

The presence of non-holonomic constraints is accounted for by a submanifold $\mathcal{A}$ of $j_1(\mathcal{V}_{n+1})$, fibred over $\mathcal{V}_{n+1}$ and referred to local fibred coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$. The imbedding $\mathcal{A} \xrightarrow{i} j_1(\mathcal{V}_{n+1})$ is locally expressed as

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \qquad i = 1, \ldots, n.$$

A section $\gamma : \mathbb{R} \to \mathcal{V}_{n+1}$ is called *admissible* if and only if there exists a section $\hat{\gamma} : \mathbb{R} \to \mathcal{A}$, called the *lift* of γ, locally described as $q^i = q^i(t), z^A = z^A(t)$ and satisfying $i \cdot \hat{\gamma} = j_1(\gamma)$. In the stated circumstance, the section $\hat{\gamma}$ too is called admissible. In coordinates, the admissibility condition reads

$$\frac{dq^i}{dt} = \psi^i(t, q^1(t), \ldots, q^n(t), z^1(t), \ldots, z^r(t)).$$

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