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New exact and numerical solutions of the (convection–)diffusion kernels on SE(3)

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ABSTRACT

We consider hypo-elliptic diffusion and convection–diffusion on $\mathbb{R}^3 \rtimes S^2$, the quotient of the Lie group of rigid body motions SE(3) in which group elements are equivalent if they are equal up to a rotation around the reference axis. We show that we can derive expressions for the convolution kernels in terms of eigenfunctions of the PDE, by extending the approach for the SE(2) case. This goes via application of the Fourier transform of the PDE in the spatial variables, yielding a second order differential operator. We show that the eigenfunctions of this operator can be expressed as (generalized) spheroidal wave functions. The same exact formulas are derived via the Fourier transform on SE(3). We solve both the evolution itself, as well as the time-integrated process that corresponds to the resolvent operator. Furthermore, we have extended a standard numerical procedure from SE(2) to SE(3) for the computation of the solution kernels, that is directly related to the exact solutions.

1. Introduction

1.1. Background and motivation

The properties of the kernels of the hypo-elliptic (convection–)diffusions on Lie groups, in particular the Euclidean motion group, have been of interest in fields such as image analysis [1–6], robotics [7] and harmonic analysis [8–11]. In [12] Mumford posed the problem of finding solutions of the kernels for the convection–diffusion process (direction process) on the roto-translation group SE(2). Subsequently, in [13] analytic approximations were provided. Several numerical approaches were provided in [1,14–16,9]. Exact solutions were derived in [17].

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In [10], a Fourier-based method is presented to compute the kernel of the hypo-elliptic heat equation on unimodular Lie groups and examples are provided for various Lie groups. Three approaches to derive the exact solutions for the heat kernel for SE(2) have been proposed in SE(2) in [18,6], of which the first is equivalent to the approach in [10] and provides a solution in terms of a Fourier series. This approach will be extended in this paper to SE(3) and the connection to the Fourier transform on SE(3) will be made. The second approach of [18,6] provides a solution in terms of rapidly decaying functions and in the third approach this series is explicitly computed in terms of Mathieu functions.

In image analysis of 2D images, both the convection–diffusion and the diffusion PDEs can be used for crossing-preserving enhancement of elongated structures, after the image is lifted from $\mathbb{R}^2$ to the group SE(2) via an invertible orientation score [19], other types of lifting operators [3,20], or using a semi-discrete variant [21]. Enhancement of such structures is important in for example retinal imaging to improve the detection of blood vessels, which is challenging due to the low contrast and noise in the images.

Recently, both the diffusion and the convection–diffusion PDE have become of interest for enhancement of diffusion-weighted Magnetic Resonance Imaging (dMRI). The diffusion orientation distribution function (ODF) or the fiber orientation distribution (FOD) function can be considered to be a function on a domain in $\mathbb{R}^3 \times S^2$, indicating at each position the estimated diffusion profile or distribution of fibers on the sphere S^2. Such functions are usually position-wise estimated from the dMRI data. The advantage of processing with these PDEs is that they induce a better alignment between local orientations and their surrounding. In [5] the 3D extension of Mumford's direction process was used enhance dMRI data with the aid of stochastic completion fields. In [22] it is shown that the convection–diffusion kernel can be used to obtain asymmetric, regularized FODs. The practical advantages of the diffusion process for regularization of dMRI data, in particular better fiber tractography results and improved connectivity measures, are given in [23–26].

Although finite-difference implementations [27] of the PDEs exist, as well as kernel approximations [28] and improvements of these [29], so far no exact expressions are known. The derivation of these exact solutions will be one of the main results of this paper. Other contributions of this work are summarized at the end of this introduction. We first provide more details on the mathematical setting, established in previous work [28].

1.2. Left-invariant convection–diffusion operators on SE(3)

Let $\tilde{U} : SE(3) \to \mathbb{R}^+$ be a square integrable function defined on the Lie group of rigid body motions $SE(3) = \mathbb{R}^3 \rtimes SO(3)$, which is the semi-direct product of the translation group $\mathbb{R}^3$ and the rotation group $SO(3)$. We consider $\tilde{U}$ to be a given input, that requires regularization or enhancement of certain features. We use a tilde throughout the paper to indicate functions and operators on the group $SE(3)$ (in contrast to functions/operators on a group quotient later in the paper, that we denote without a tilde). A possible approach is to use particular evolution equations that are special cases of the following general evolution process:

$$\begin{cases} \dfrac{\partial \tilde{W}(g,t)}{\partial t} = \tilde{Q}\tilde{W}(g,t), & \text{for all } g \in \mathrm{SE}(3),\ t \geq 0, \\ \tilde{W}(g,0) = \tilde{U}(g), & \text{for all } g \in \mathrm{SE}(3). \end{cases} \tag{1}$$

Here $\tilde{Q}$ is the generator of the evolution on the group, where we restrict ourselves to generators such that the evolution becomes a linear, second order convection–diffusion process. Moreover, $\tilde{Q}$ should be a left-invariant operator and is composed of the left-invariant differential operators $\underline{\mathcal{A}} = (\mathcal{A}_1, \ldots, \mathcal{A}_6)$. These vector fields are obtained via a pushforward $\mathcal{A}_i|_g = (L_g)_* A_i$, where $\{A_i\}_{i=1}^6$ is a choice of basis in the tangent space $T_e(SE(3))$ at the unity element $e = (0, I)$, where A_1, A_2, A_3 are spatial tangent vectors and A_4, A_5, A_6 are rotational tangent vectors. Here L_g is the left-multiplication $L_g q = gq$.