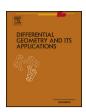
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## Differential Geometry and its Applications





# Examples of flag-wise positively curved spaces



Ming Xu<sup>1</sup>

College of Mathematics, Tianjin Normal University, Tianjin 300387, PR China

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#### ABSTRACT

A Finsler space (M, F) is called flag-wise positively curved, if for any  $x \in M$  and any tangent plane  $\mathbf{P} \subset T_x M$ , we can find a nonzero vector  $y \in \mathbf{P}$ , such that the flag curvature  $K^F(x, y, \mathbf{P}) > 0$ . Though compact positively curved spaces are very rare in both Riemannian and Finsler geometry, flag-wise positively curved metrics should be easy to be found. A generic Finslerian perturbation for a non-negatively curved homogeneous metric may have a big chance to produce flag-wise positively curved metrics. This observation leads our discovery of these metrics on many compact manifolds. First we prove any Lie group G such that its Lie algebra  $\mathfrak g$  is compact non-Abelian and  $\dim \mathfrak{c}(\mathfrak{g}) \leq 1$  admits flag-wise positively curved left invariant Finsler metrics. Similar techniques can be applied to our exploration for more general compact coset spaces. We will prove, whenever G/H is a compact coset space with a finite fundamental group, G/H and  $S^1 \times G/H$  admit flag-wise positively curved Finsler metrics. This provides abundant examples for this type of metrics, which are not homogeneous in general. These examples implies a significant difference between the flag-wise positively curved condition and the positively curved condition, even though they are reduced to the same one in Riemannian geometry.

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#### 1. Introduction

A Finsler metric on a smooth manifold M is a continuous function  $F:TM\to [0,+\infty)$  satisfying the following conditions:

- (1) F is a positive smooth function on the slit tangent bundle  $TM \setminus 0$ ;
- (2)  $F(x, \lambda y) = \lambda F(x, y)$  for any  $x \in M$ ,  $y \in T_x M$ , and  $\lambda \ge 0$ ;
- (3) For any standard local coordinates  $x = (x^i)$  and  $y = y^i \partial_{x^i}$  on TM, the Hessian matrix

$$(g_{ij}^F(x,y)) = (\frac{1}{2}[F^2(x,y)]_{y^iy^j})$$

E-mail address: mgmgmgxu@163.com.

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is positive definite for any nonzero  $y \in T_xM$ , i.e. it defines an inner product

$$\langle u, v \rangle_y^F = \frac{1}{2} \frac{d^2}{dsdt} F^2(y + su + tv)|_{s=t=0} = g_{ij}^F(x, y) u^i v^j$$

for any  $u = u^i \partial_{x^i}$  and  $v = v^j \partial_{x^j}$  in  $T_x M$ .

We call (M, F) a Finsler space or a Finsler manifold. The restriction of the Finsler metric to a tangent space is called a *Minkowski norm*. Minkowski norm can also be defined on any real vector space by similar conditions as (1)–(3), see [1] and [4].

In Finsler geometry, flag curvature is the natural generalization for sectional curvature in Riemannian geometry. But the flag curvature  $K^F(x, y, \mathbf{P})$  is a much more localized geometric quantity in the sense that it depends on tangent plane  $\mathbf{P} \in T_x M$  as well as the nonzero base vector  $y \in \mathbf{P}$ , see Section 2 below. This inspires us to define the following generalization for the positively curved condition in Finsler geometry [10].

**Definition 1.1.** Let (M, F) be a Finsler space. We say a tangent plane  $\mathbf{P} \subset T_x M$  satisfies the (FP) condition if there exists a nonzero vector  $y \in T_x M$  such that the flag curvature  $K^F(x, y, \mathbf{P}) > 0$ . We say (M, F) satisfies the (FP) condition or it is flag-wise positively curved if all its tangent planes satisfy the (FP) condition.

In [10], we have found many compact coset spaces which admit non-negatively and flag-wise positively curved homogeneous Finsler metrics, but no positively curved homogeneous Finsler metrics. If concerning the flag-wise positively curved condition alone, we will have much more chance finding new metrics of this type. We can start with a canonical homogeneous metric of non-negative curvature, for example, bi-invariant metrics on Lie groups (this will require G to be quasi-compact, i.e.  $\mathfrak{g} = \text{Lie}(G)$  is compact), and normal homogeneous metrics on compact homogeneous space [3]. Then generic Finslerian perturbations may produce flag-wise positively curved Finsler spaces.

In this paper, we will justify this observation. First we will prove the following main theorem, which gives a positive answer to Problem 4.4 in [10].

**Theorem 1.2.** Any Lie group G such that  $\text{Lie}(G) = \mathfrak{g}$  is a compact non-Abelian Lie algebra with  $\dim \mathfrak{c}(\mathfrak{g}) \leq 1$  admits a flag-wise positively curved left invariant Finsler metric.

As in Section 4 of [10], where we prove Theorem 1.2 when rkg = 2, the construction for the metric is based on the Killing navigation technique, but we need a more complicated gluing process here.

Using the homogeneous flag curvature formula [11,5], it is not hard to see any Lie group G with  $\dim \mathfrak{c}(\mathfrak{g}) > 1$  does not admit flag-wise positively curved left invariant Finsler metrics. So Theorem 1.2 indicates the exact obstacle when  $\mathfrak{g}$  is compact. When  $\mathfrak{g}$  is not compact, it is still unknown if we can find more examples. A more interesting problem is

**Problem 1.3.** Classify all the Lie groups G which admit left invariant non-negatively and flag-wise positively curved Finsler metrics.

It is closely related to Problem 4.1 and Problem 4.2 in [10]. We guess the group G indicated by Problem 1.3 must be SO(3) or SU(2). Until now, we only know dim  $\mathfrak{c}(\mathfrak{g}) < 2$ ,  $\mathfrak{g}$  must be unimodular, and it can not be nilpotent [6].

With the similar method, we can even prove

**Theorem 1.4.** For any compact coset space G/H with a finite fundamental group, we can find flag-wise positively curved Finsler metrics on G/H and  $S^1 \times G/H$ .

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