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Examples of flag-wise positively curved spaces

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ABSTRACT

A Finsler space (M, F) is called flag-wise positively curved, if for any $x \in M$ and any tangent plane $\mathbf{P} \subset T_xM$, we can find a nonzero vector $y \in \mathbf{P}$, such that the flag curvature $K^F(x, y, \mathbf{P}) > 0$. Though compact positively curved spaces are very rare in both Riemannian and Finsler geometry, flag-wise positively curved metrics should be easy to be found. A generic Finslerian perturbation for a non-negatively curved homogeneous metric may have a big chance to produce flag-wise positively curved metrics. This observation leads our discovery of these metrics on many compact manifolds. First we prove any Lie group G such that its Lie algebra $\mathfrak{g}$ is compact non-Abelian and $\dim \mathfrak{c}(\mathfrak{g}) \leq 1$ admits flag-wise positively curved left invariant Finsler metrics. Similar techniques can be applied to our exploration for more general compact coset spaces. We will prove, whenever G/H is a compact coset space with a finite fundamental group, G/H and $S^1 \times G/H$ admit flag-wise positively curved Finsler metrics. This provides abundant examples for this type of metrics, which are not homogeneous in general. These examples implies a significant difference between the flag-wise positively curved condition and the positively curved condition, even though they are reduced to the same one in Riemannian geometry.

1. Introduction

A *Finsler metric* on a smooth manifold M is a continuous function $F : TM \to [0, +\infty)$ satisfying the following conditions:

(1) F is a positive smooth function on the slit tangent bundle $TM\backslash 0$;
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$, and $\lambda \geq 0$;
(3) For any *standard local coordinates* $x = (x^i)$ and $y = y^i \partial_{x^i}$ on TM, the Hessian matrix

$$(g_{ij}^F(x, y)) = (\frac{1}{2}[F^2(x, y)]_{y^i y^j})$$

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is positive definite for any nonzero $y \in T_xM$, i.e. it defines an inner product

$$\langle u, v\rangle_y^F = \frac{1}{2}\frac{d^2}{dsdt}F^2(y + su + tv)|_{s=t=0} = g_{ij}^F(x, y)u^i v^j$$

for any $u = u^i\partial_{x^i}$ and $v = v^j\partial_{x^j}$ in T_xM.

We call (M, F) a *Finsler space* or a *Finsler manifold*. The restriction of the Finsler metric to a tangent space is called a *Minkowski norm*. Minkowski norm can also be defined on any real vector space by similar conditions as (1)–(3), see [1] and [4].

In Finsler geometry, flag curvature is the natural generalization for sectional curvature in Riemannian geometry. But the flag curvature $K^F(x, y, \mathbf{P})$ is a much more localized geometric quantity in the sense that it depends on tangent plane $\mathbf{P} \in T_xM$ as well as the nonzero base vector $y \in \mathbf{P}$, see Section 2 below. This inspires us to define the following generalization for the positively curved condition in Finsler geometry [10].

Definition 1.1. Let (M, F) be a Finsler space. We say a tangent plane $\mathbf{P} \subset T_xM$ satisfies the (FP) condition if there exists a nonzero vector $y \in T_xM$ such that the flag curvature $K^F(x, y, \mathbf{P}) > 0$. We say (M, F) satisfies the (FP) condition or it is flag-wise positively curved if all its tangent planes satisfy the (FP) condition.

In [10], we have found many compact coset spaces which admit non-negatively and flag-wise positively curved homogeneous Finsler metrics, but no positively curved homogeneous Finsler metrics. If concerning the flag-wise positively curved condition alone, we will have much more chance finding new metrics of this type. We can start with a canonical homogeneous metric of non-negative curvature, for example, bi-invariant metrics on Lie groups (this will require G to be quasi-compact, i.e. $\mathfrak{g} = \mathrm{Lie}(G)$ is compact), and normal homogeneous metrics on compact homogeneous space [3]. Then generic Finslerian perturbations may produce flag-wise positively curved Finsler spaces.

In this paper, we will justify this observation. First we will prove the following main theorem, which gives a positive answer to Problem 4.4 in [10].

Theorem 1.2. *Any Lie group G such that $\mathrm{Lie}(G) = \mathfrak{g}$ is a compact non-Abelian Lie algebra with $\dim \mathfrak{c}(\mathfrak{g}) \leq 1$ admits a flag-wise positively curved left invariant Finsler metric.*

As in Section 4 of [10], where we prove Theorem 1.2 when $\mathrm{rk}\mathfrak{g} = 2$, the construction for the metric is based on the Killing navigation technique, but we need a more complicated gluing process here.

Using the homogeneous flag curvature formula [11,5], it is not hard to see any Lie group G with $\dim \mathfrak{c}(\mathfrak{g}) > 1$ does not admit flag-wise positively curved left invariant Finsler metrics. So Theorem 1.2 indicates the exact obstacle when $\mathfrak{g}$ is compact. When $\mathfrak{g}$ is not compact, it is still unknown if we can find more examples. A more interesting problem is

Problem 1.3. *Classify all the Lie groups G which admit left invariant non-negatively and flag-wise positively curved Finsler metrics.*

It is closely related to Problem 4.1 and Problem 4.2 in [10]. We guess the group G indicated by Problem 1.3 must be SO(3) or SU(2). Until now, we only know $\dim \mathfrak{c}(\mathfrak{g}) < 2$, $\mathfrak{g}$ must be unimodular, and it can not be nilpotent [6].

With the similar method, we can even prove

Theorem 1.4. *For any compact coset space G/H with a finite fundamental group, we can find flag-wise positively curved Finsler metrics on G/H and $S^1 \times G/H$.*