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Mixed type surfaces with bounded mean curvature in 3-dimensional space-times ☆

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ABSTRACT

In this paper, we shall prove that space-like surfaces with bounded mean curvature functions in real analytic Lorentzian 3-manifolds can change their causality to time-like surfaces only if the mean curvature functions tend to zero. Moreover, we shall show the existence of such surfaces with non-vanishing mean curvature and investigate their properties.

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1. Introduction

We say that a connected surface immersed in a Lorentzian 3-manifold (M^3, g) is of *mixed type* if both the space-like and time-like point sets are non-empty. In general, the mean curvature of such surfaces diverges: for example, the graph of a smooth function $t = f(x, y)$ in the Lorentz–Minkowski space-time $(\boldsymbol{R}^3_1; t, x, y)$ gives a space-like (resp. time-like) surface if $B > 0$ (resp. $B < 0$), where

$$B := 1 - f_x^2 - f_y^2. \tag{1.1}$$

In this situation, the unit normal vector is given by

$$\nu = \frac{1}{\sqrt{|B|}}(1, f_x, f_y), \tag{1.2}$$

and the mean curvature function is computed as

$$H = \frac{\left(f_x^2 - 1\right) f_{yy} - 2f_x f_y f_{xy} + \left(f_y^2 - 1\right) f_{xx}}{2|B(x,y)|^{3/2}}, \tag{1.3}$$

which is unbounded around the set $\{B(x, y) = 0\}$, in general.

On the other hand, several zero mean curvature surfaces of mixed type in $\boldsymbol{R}^3_1$ were found in [2–5,7–10]. Moreover, such examples can be found in other space-times: in fact, a zero mean curvature surface of mixed type in the de Sitter 3-space (resp. in the anti-de Sitter 3-space) is given in this paper (cf. Example 2.7 and Example 2.8). It is known that zero mean curvature surfaces in $\boldsymbol{R}^3_1$ change types across their fold singularities, except for the special case as in [2]. On the other hand, in [6], it was shown that space-like non-zero constant mean curvature surfaces do not admit fold singularities, which suggests that space-like non-zero constant mean curvature surfaces never change types. More precisely, the following questions naturally arise:

(a) *Is there a mixed type surface with non-zero constant mean curvature?*
(b) *Is there a mixed type surface whose mean curvature vector field is smooth and does not vanish along the curve of type change?*

In this paper, we show that the answer to Question (a) is negative. This is a consequence of the following assertion:

Theorem 1.1. *Let U be a connected domain in $\boldsymbol{R}^2$, and $f : U \to (M^3, g)$ a real analytic immersion into an oriented real analytic Lorentzian manifold (M^3, g). We denote by U_+ (resp. U_-) the set of points where f is space-like (resp. time-like). Suppose that U_+, U_- are both non-empty, and the mean curvature function H on $U_+ \cup U_-$ is bounded. Then for each $p \in \overline{U_+} \cap \overline{U_-}$, there exists a sequence $\{p_n\}_{n=1,2,3,..}$ in U_+ (resp. U_-) converging to p such that $\lim_{n\to\infty} H(p_n) = 0$, where $\overline{U_+}, \overline{U_-}$ are the closures of U_+, U_- in U.*

It should be remarked that the higher dimensional version of the theorem holds (see Remark 2.2). There exist space-like and time-like constant mean curvature immersions in $\boldsymbol{R}^3_1$ which are not of mixed type although their induced metrics degenerate along certain smooth curves (cf. Examples 2.4 and 2.5 in Section 2). Also, there are similar examples of space-like constant mean curvature one surfaces in the de Sitter 3-space S^3_1 with singularities which are not of mixed type ([1]). The existence of such examples implies that we cannot drop the assumption that both U_+, U_- are non-empty. The proof of Theorem 1.1 is given in Section 2.